The billiards problem

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Abstract

In order to see objects, light must shine on them and reflect off. If you were to shine a light at a mirror, the light would bounce back. In this same manner, billiards balls bounce around a billiards table in mathematical billiards. This paper will examine some major findings from Andrew M. Baxter and Ronald Umble's paper, titled *Periodic Orbits for Billiards on an Equilateral Triangle,* and will apply some of their findings to an example problem. The focus will be aspects of period orbits on billiards tables in the shape of an equilateral triangle.

1. Introduction

Light is essential for us to see. In order to see the world around us, light must be present and reflect off of objects. If one was to shine a flashlight at a mirror, the light would reflect back. The angle that light hits the surface at is know as the angle of incidence and the angle that the light reflects at is known as the angle of reflection. The reflection of light is often described as light bouncing off of a surface. This happens because light beams reflect off of flat surfaces in the same manner that billiards balls bounce off the rim of the table. This is exactly the same way a billiards ball bounces off side of the table. Billiards is a popular game with people of all ages. The game of billiards originated in Europe and the goal is for one player to hit their cue ball at the opposing player's cue ball and the object ball at the same time. Traditional billiards tables do not have pockets. The term billiards has morphed into an umbrella term in the United States and can be used to mean any game involving a cue stick. The most well known billiards game in America is pool, which is played on a table with pockets shaped like a rectangle. When playing, the objective is to hit the other player's balls into the pockets before they do so to your balls. Billiards balls can take many different paths and their path is determined three things: the starting position of the ball, the direction the ball is traveling in, and the speed the ball is moving at. It is important to make the distinction between physical billiards and mathematical billiards. The spin of the ball is also a factor in physical billiards. In mathematical billiards, the spin of the ball is not really considered.

This paper will examine how billiards balls move around the table and the types of paths they will take. This paper will focus on periodic orbits, or orbits that repeat. It will look specifically at how billiards balls orbit around an equilateral triangle and will also use an unfolding technique to examine the orbits made. This paper will also look closely at a paper entitled *Periodic Orbits for Billiards on an Equilateral Triangle* by Andrew M. Baxter and

Ronald Umble in 20008 [1] and try to see if some of their findings hold for an example problem.

2. Orbits

There is an extensive history of mathematical work done with billiards. One paper, titled *Periodic Orbits for Billiards on an Equilateral Triangle* by Andrew M. Baxter and Ronald Umble, looked at the idea of unfolding periodic orbits on equilateral triangles and some applications that this has and this paper is trying to examine the ideas that are presented in Baxter and Umble's paper and apply some of them to an example problem.

To begin, let us get a basis with some terminology and simple ideas. For our purposes, we are going to assume that we have a frictionless table, in the shape of an equilateral triangle. This will allow the billiards balls to continue moving indefinitely without a loss of speed. The *trajectory* of a billiard ball is the path that it takes around the table. As stated in the introduction, this is determined by three things: the initial position of the ball, the direction the ball is traveling in, and the speed it's going at. Referring back to our light example above, the *angle of incidence* is the angle in which the ball strikes the side of the table, also known as the bumper. The *angle of reflection* is the angle that the ball bounces off the bumper at. The physics Law of Reflection lets us assume that the angle of incidence equals the angle of reflection. Since the table is frictionless, the ball will continue along its trajectory with the same speed unless it hits a vertex. In our case, the vertex would have to be a pocket. That is the only way that the ball would stop moving is if it hit a pocket and escaped from the table. There are three types of trajectories. *Periodic trajectories* are trajectories continue along the same path, which repeats, with the same speed and direction infinitely. Figure 1 shows a periodic trajectory on the left. The other two options are nonperiodic, meaning they do not repeat. *Infinite trajectories* are paths that continue forever where the billiard does not return to its initial position; the path is unique and no matter how many times the ball reflects of the sides, the path will not repeat itself. It is difficult to show a visual image of an infinite trajectory because the path will never end. Figure 1 shows a partial infinite trajectory in the center image. *Singular trajectories* are ones that end in a vertex. Figure 1 shows an example of a singular trajectory in the image on the right.

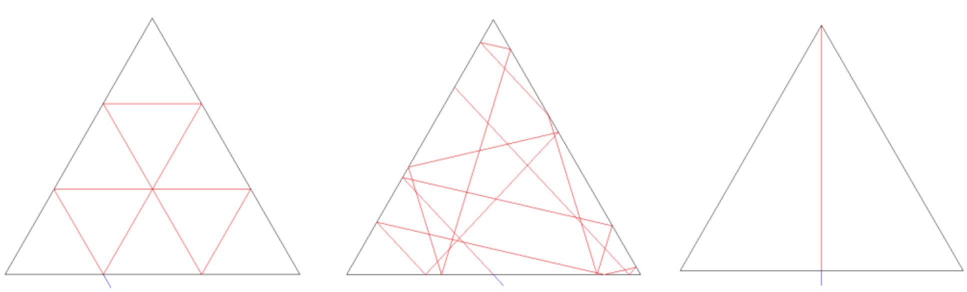

Figure 1. *Examples of period, infinite, and singular trajectories*

Billiards tables can be different shapes, but all are closed sided figures with straight sides and thus can be thought of as polygons. Since the table itself is flat, it can be described as a plane region. Thus, a billiard table can be described as a plane region, referred to as Ω, that is bounded by a polygon, π. A *nonsingular trajectory* is one that is either infinite or periodic and does not end in a vertex. The nonsingular trajectory can be defined as a piecewise linear constant speed curve $\alpha : \mathbb{R} \rightarrow \Omega$, where $\alpha(t)$ denotes the position of the ball at time t. An *orbit* is a nonsingular trajectory that has been restricted to a closed interval, meaning that you only examine the trajectory (that would normally continue indefinitely) over a fixed period of time. A nonsingular trajectory α is *periodic*, meaning it is not infinite trajectory and the billiard repeats its orbit from its initial position with its initial velocity indefinitely, if $\alpha(a + t) = \alpha(b + t)$ for some $a < b$ and all $t \in \mathbb{R}$. The restriction of the nonsingular trajectory to [a, b] is a periodic orbit. *Periodic orbits* retrace their orbits indefinitely, depending on the interval. In fact, a periodic orbit will retrace its path exactly $n \geq 1$ times. An orbit can be described as *primitive* if n is equal to 1, which means the closed interval allows the orbit to go around its orbit exactly once and appears to stop as soon as it returns to its initial point because we are looking at the orbit on a closed interval and the interval has reached its higher bound. If $n > 1$, then the orbit is an *n-fold iterate*. If $n = 1$, which means the orbit is primitive, then α^n denotes the orbit's n-fold iterate. The *period* of a periodic orbit is the number of times the billiard ball strikes the bumper during its orbit. For example, if a primitive periodic orbit strikes the wall 6 times before returning to its initial position then its period is 6. If a periodic orbit retraces it's path two times, i.e. n = 2, and it hits the sides 6 times each time it travels its path, then the period would be 12. Keep in mind that the images of a primitive orbit of period 6 and an 2-fold iterate of period 12 will be the same. Thus, if α is a primitive orbit that has period k, then the orbit α^n has period kn [1].

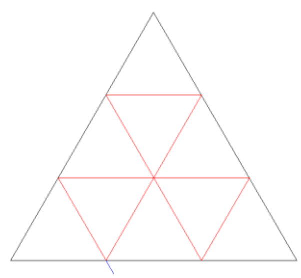

Figure 2. *Primitive periodic orbit with period 6 and a 2-fold iterate with period 12*

3. History

It is first important to consider the existence of period orbits. Giulio Fagnano was one of the first people to look at periodic orbits. Fagnano is originally from Italy and grew up in one of the most prominent families there. Fagnano viewed mathematics as a hobby and taught himself what he knew. He looked into new methods of solving equations of degrees two, three, and four. He also improved the work done on complex numbers. He is best known for his work with triangles. He also looked at elliptic integrals and this work eventually led to elliptic functions [4]. Fagnano's work with elliptic functions was the basis for Euler's addition formula for elliptic integrals. These were significant contributions to the field of mathematics and he is internationally recognized. He received many honors during his time, including the title of count, election to the Royal Society of London, election to the Berlin Academy of Sciences and proposed election to the Paris Academie des Sciences. Fagnano discovered the first examples of periodic orbits in 1745. Fagnano discovered a periodic orbit of period 3 on an acute triangle which is now known as the "Fagnano orbit". This orbit was found to be the triangle of least perimeter inscribed in a given acute triangle, and is known as "Fagnano's problem". This problem was solved by an orthic triangle. An *orthic triangle* is a triangle that joins the feet of the altitudes in a given triangle. Since the angles of an orthic triangle are bisected by the heights of the triangle in which it is inscribed, orthic triangles are periodic trajectories of period 3 [4].

Harold Scott McDonald Coxeter, known as Donald Coxeter (from his third name), also worked with orthic triangles and was able to prove that an orthic triangle is a periodic orbit on the triangle it is inscribed in using an "unfolding" technique that will be discussed in detail later in this paper. Coxeter lived much later than Fagnano and helped to bring Fagnano's work into the 20th century. He was educated at the University of Cambridge and worked at Princeton University and the University of Toronto. Coxeter did most of his work in the field of geometry, specifically making contributions to the theory of polytopes, non-euclidian geometry, group theory, and combinatorics. Coxeter groups are discrete reflection groups and they lead to tessellations. In 1934, Coxeter was able to classify all spherical and euclidian Coxeter groups [4]. Coxeter was motivated by the beauty of mathematics and he received nine honorary doctorates and is a Fellow of two royal societies. Coxeter is widely published; he has written 12 books and 167 articles. Coxeter was also interested in music and had originally dreamed of becoming a composer. He believes that being vegetarian, following a strict exercise schedule, and loving what he does have helped his life to be long and successful. Coxeter died in 2003 at the age of 96. Coxeter applied tessellations to orthic triangles to prove they are periodic. He used an unfolding technique that will be explained in detail later in this paper.

There have been a few other people that made significant contributions to the field of mathematics and topics mentioned in this paper. Coxeter gives credit to H. A. Schwarz for the unfolding technique used in the proof that orthic triangles are periodic orbits. Frank and F.V. Morely worked to see if they could apply what Schwarz did with triangles to other polygons and they found that they can unfold odd-sided polygons in a similar manner. Masur also did work with polygons on billiards tables in the late 1900s and found that there are infinitely many periodic orbits with distinct periods that come from rational polygons. A *rational polygon* is one where the interior angles, or angles between each pair of sides, are rational multiples of π [4].

4. Baxter & Umble (2008)

The paper, *Periodic Orbits for Billiards on an Equilateral Triangle* by Andrew M. Baxter and Ronald Umble looked at periodic orbits. These orbits can take place on a variety of shapes. For this paper, we are interested in looking specifically at periodic orbits on an equilateral triangle. Baxter and Umble were interested in the problem of finding, classifying, and counting the classes of periodic orbits on an equilateral triangle. In order to consider some of their specific findings, let us consider an equilateral triangle, $\triangle ABC$. Orbits can take many different paths on their trajectory inside the triangle, however there are some rules governing the angles of trajectory. This leads us to proposition 1. Baxter and Umble specifically found that:

Proposition 1. *Every nonsingular trajectory strikes some side of △ABC with an angle of incidence in the range $30° \leq \theta \leq 60°$ [1].*

This simply means that for any orbit that is either periodic or infinite, somewhere on its path the orbit will have an angle of incidence between the range $30° \leq \theta \leq 60°$. For a full proof of this proposition, see Baxter and Umble's full paper [1].

Now let us consider a periodic orbit with period n on △ABC. Let us call this orbit α. Rotate △ABC so that line segment BC is horizontal and is the base of the triangle. We can use proposition 1 with periodic orbits in the same way we used it with nonsingular trajectories before. Start with a point P, that falls on line segment BC and has an angle of incidence that falls in the range $30° \leq \theta \leq 60°$. Our orbit α will begin and end at point P since α is periodic. A *tessellation* is a tiling of a plane by repetition a figure over and over with no gaps or overlaps. We can tessellate our plane with copies of the equilateral triangle △ABC. Call this tessellation T. When doing this, we want to keep parallel edges horizontal. Place our initial triangle, △ABC, somewhere in the tessellation so that our base, line segment BC, falls on one of the horizontal parallel edges. Our next step is to label the line segments connecting each strike point of α in order as α_1, α_2, $\alpha_{3,}$ … , α_n. It is now possible for us to literally unfold the periodic orbit α onto our tessellated plane, T, to show the orbit in a straight path. Start with our starting point, P. Then the next strike point, connected by $\alpha_{1,}$ can be labeled P_1. Also label the third strike point, connected by $\alpha_{2,}$ as P_2 . If we call the side that P_1 is on $s_{1,}$ we can reflect △ABC across side 1 and α_2 will now be a collinear line segment with α_1 is the adjacent triangle in our tessellation. In this manner, we can continue to unfold an entire period orbit to form a straight line. Label Q and the final termination point of the unfolded orbit. Then we can call <u>PQ</u> an unfolding of α and to θ_n as its representation angle [1].

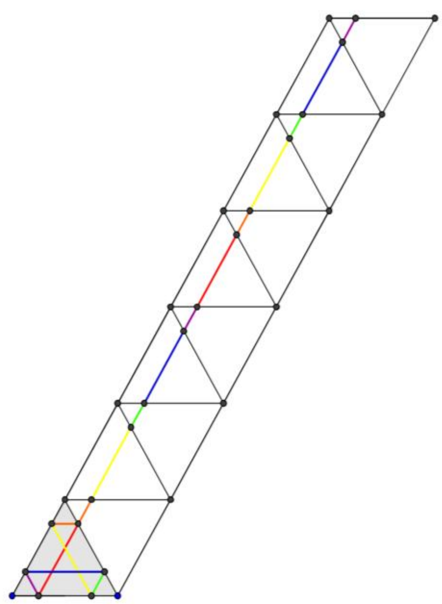

Figure 3. *Example of an unfolding of period 12*

This leads us to wonder if unfoldings can happen on shapes other than an equilateral triangle. In order for this to work, the polygon used must be able to tessellate the plane completely with no overlaps or gaps. There are only a few polygons where this will work. In order for a polygon to tessellate the plane, the angles of the polygon must be factors of 360. This is the case because there are 360 degrees in a circle and the vertices of the polygons must completely fill a circle in order to have no gaps when tiling the plane. The factors of 360 are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, and 360. However, not all of these factors make sense for angle measures of a polygon. A polygon could have angle measures of 1°, 2°, 3°, 4°, 5°, 6°, 8°, 9°, 10°, 12°, 15°, 18°, 20°, 24°, 30°, 36°, 40°, 45°, 60°, 72°, 90°, and 120°. In order for the unfolding technique to work, each shape need to have sides and angles that are congruent so that the reflection could take place on any side. This means that only regular polygons, ones with every side and angle congruent, that tile the plane can be used in the unfolding. This limits the polygons to equilateral triangles, 45°- 45°- 90° triangles, 30°- 30°- 120° triangles, squares, hexagons. Any of these polygons could be used to tessellate the plane and an unfolding of a periodic orbit within one of the shapes could be done. Figure 4 shows an example of an unfolding of a periodic orbit on a square.

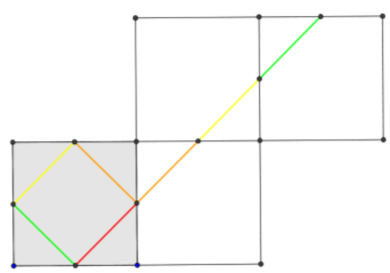

Figure 4. *Example of an unfolding on a square*

There are orbits of many different periods. Regardless of the period, a periodic orbit will consist of at most three incidence angles. It can hit the bumpers more than three times, but there will only be at most three distinct angle measures. This leads us to proposition 2, another proposition that Baxter and Umble (2008) specifically found. For a full proof of this finding please refer to their paper [1].

Proposition 2. *A periodic orbit strikes the sides of △ABC with at most three incidence angles, exactly one of which lies in the range* $30° \leq \theta \leq 60°$. *In fact, exactly one of the following holds:*

(1) All incidence angles measure 60°.

(2) There are exactly two distinct incidence angles measuring 30° and 90°.

(3) There are exactly three distinct incidence angles Φ, θ, Ψ *such that*

$$0° < \Phi < 30° < \theta < 60° < \Psi < 90° \text{ [1]}$$

If (1) is true and the first angle of incidence, θ_1, is 60° we can follow the path of the orbit and use some of the work done in proposition 1 to determine that every angle of incidence must be 60°. This leaves an orbit that sections △ABC into many smaller equilateral triangles. Figure 5 shows an example of a period 3 orbit that has all angles equal to 60°.

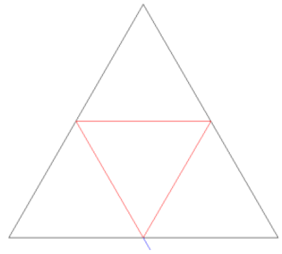

Figure 5. *Orbit with all angles* 60°

If (2) is the one that holds then the periodic orbit is oscillating and looks like two lines coming out of the initial point.

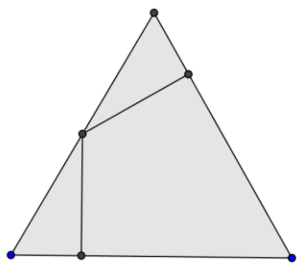

Figure 6. *Orbit with angle 1 at 30°and angle 2 at 90°*

If (3) is the part of the proposition that is true, then we are working with an orbit that is more interesting in my opinion. This orbit will form a pattern that cannot be predicted. This orbit can still be unfolded which can help determine the period of the orbit.

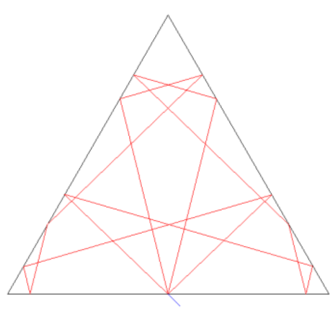

Figure 7. *Orbit with* $0° < \Phi < 30° < \theta < 60° < \Psi < 90°$

We will use these items about the angles of the orbits as we continue exploring what happens when an orbit is unfolded. Some facts can be noticed about unfoldings of orbits. Parallel is when two lines will extend and never cross. This idea can be applied to unfoldings of periodic orbits. Baxter and Umble specifically found the following. For a full proof of this finding please refer to their paper [1].

Corollary 1. *Any two unfoldings of a periodic orbit are paralle [1].*

Two unfoldings would create the image of two adjacent equilateral triangles which would form a rhombus and thus create parallel edges. This finding will come in useful in classifying orbits as either even or odd. The following theorem is another significant finding that Baxter and Umble found. For a full proof of this finding please refer to their paper [1].

Theorem 1. *If an unfolding of a periodic orbit α terminates on a horizontal edge of T, then α has even period[1].*

If an unfolding of an orbit terminates on a horizontal edge of the tessellated plane T, then we know that there are parallel lines, since we have structured our tessellated plane to have the parallel edges be collinear with the horizontal lines. From corollary 1, we know that two unfoldings will create parallel edges. In order for the unfolding of a periodic orbit to terminate on a horizontal edge, there must be parallel lines and thus we must have an even number of unfoldings. Each unfolding represents one strike point and the period of a periodic orbit is determined by the number of strike points. Thus, if we have an even number of unfoldings, we must also have an even number of strike points which would make the period even. Figure 8 shows an example of a tessellated plane and it is shown that the horizontal edges are also parallel.

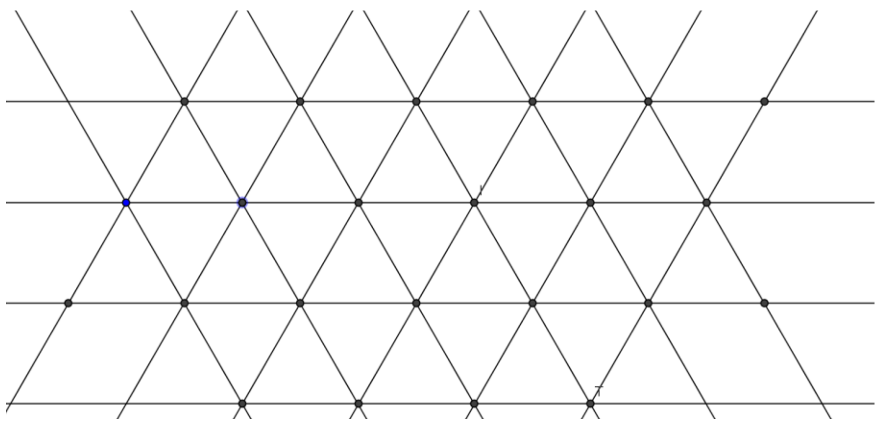

Figure 8. *An example of a tessellated plane with parallel horizontal edges*

5. An Example: Orbits of Period 12 and Period 14

Now we are going to look specifically at orbits of period 12 and period 14 and see if some of the findings that Baxter and Umble found do hold true for these orbits. It is important to remember that we are not considering all of their findings, we are only looking a specific ones.

A. Orbits of Period 12

Let us first look at the orbits of period 12. There are two distinct classes of orbits that are period twelve and none of these orbits are primitive. The two classes are 2-fold iterates and 4-fold iterates. Figure 9 shows an example of a 2-fold iterate (left) and a 4-fold iterate (right)

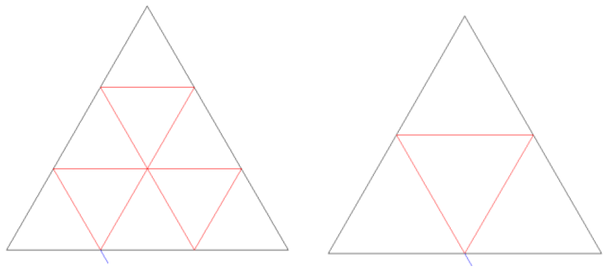

Figure 9. *Example of a 2-fold iterate and a 4-fold iterate of period 12*

We can also see that both proposition 1 and proposition 2 hold true for these classes of orbits. Let us look at an example of one 2-fold iterate to illustrate our point. We will find that there is indeed an angle that falls in the range 30° ≤ θ ≤ 60°. In fact, this angle is exactly 60°. This also means that we have a case 1 situation for proposition 2 and every angle of incidence in the orbits of period 12 will be 60°. Figure 10 shows an illustration with the 60° angle marked in.

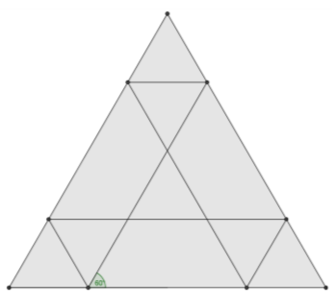

Figure 10. *Period 12 orbit with an angle of 60°.*

We can also look at a sample unfolding of this orbit to see what that would look like. We will note that the orbit does terminate on a horizontal edge of the tessellated plane so we know the period is even. Figure 3 shows the image of an unfolding of period 12.

B. Orbits of Period 14

We can contrast our example of period 12 with orbits of period 14. There is only one distinct class of orbits of period 14 and these orbits are primitive. That means that the ball will trace its trajectory once before returning to its initial position with period 14. We can also look at proposition 1 and determine that there is indeed an angle that lies in the range 30° ≤ θ ≤ 60°. There is an angle that measures 43.9°. We can also see that we have a our orbit can be classified as case 3 for proposition 2 since Φ = 16.1°, θ = 43.9°, and Ψ = 76.1°. Figure 11 shows the image of a period 14 orbit with the angles marked in.

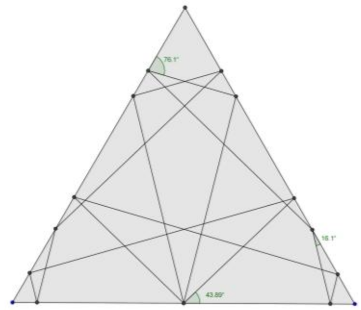

Figure 11. *Orbit of period 14 with Φ = 16.1°, θ = 43.9°, and Ψ = 76.1°*

We can also look at a sample unfolding of this orbit to see what that would look like. We will note that the orbit does terminate on a horizontal edge of the tessellated plane so we know the period is even. Figure 12 shows the image of an unfolding of period 14.

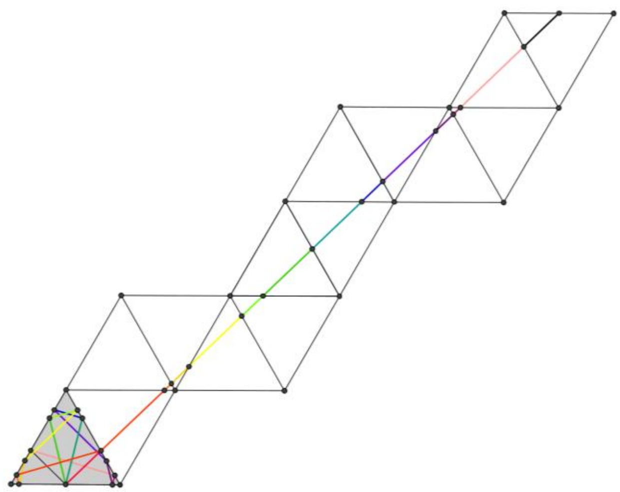

Figure 12. *Example of an unfolding of period 14.*

6. Rhombic Coordinates

Rhombic coordinates are needed in order to actually count the number of classes of orbits on a periodic triangle. The rhombic coordinate system is similar to the standard Cartesian coordinate system except that instead of squares, the coordinate system is based on a rhombus so the whole system is appearing like it is at a slant. This coordinate system is created from the tessellation of equilateral triangles; two triangles form a rhombus. In the coordinate plane however, all of the lines are left in from the triangles. See Baxter and Umble's paper for the full proof of how to count the classes of period orbits on a given equilateral triangle. Figure 13 shows the rhombic coordinate system, using O to denote the origin.

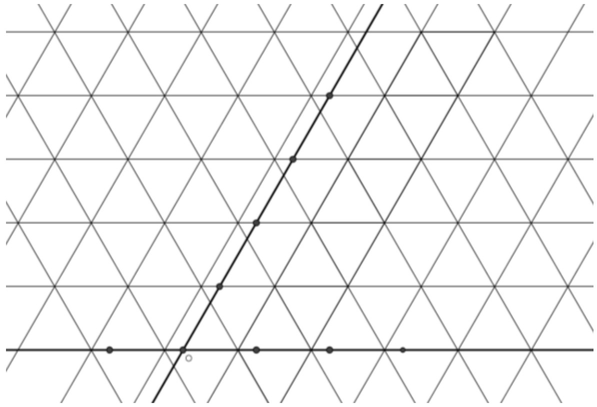

Figure 13. *Rhombic Coordinate System*

We can then place our unfoldings of period 12 and period 14 on the rhombic coordinate system. The terminating coordinate will be a point, in the form (*x, y*), and we know from Baxter and Umble's paper that the period of the orbit is going to be two times the number of rhombic tiles, or $2(x + y)$, where x and y are the same x and y in the coordinate pair (x, y). For our example of period 12 and period 14, the period 12 orbit would terminate at the point (0, 12) and the period 14 orbit would terminate at point (2, 5). We can put these two coordinate pairs into our equation $2(x + y)$ and see that it does indeed equal the period. Figure 14 shows an example of the period 14 orbit unfolded on the rhombic coordinate system with the terminating point marked.

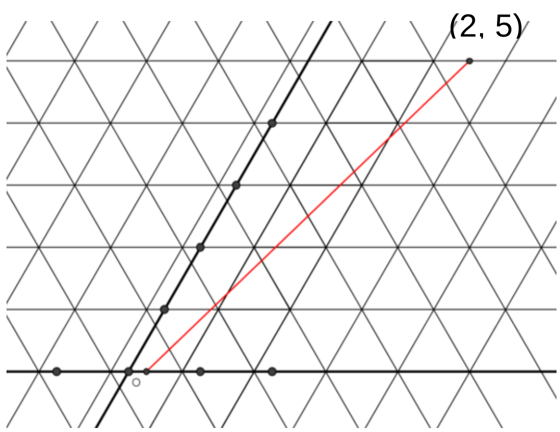

Figure 14. *Period 14 orbit unfolded on Rhombic Coordinate System*

7. Final Remarks

A final remark about this paper is that there is a purpose to looking at non primitive orbits. There is an equation that Baxter and Umble found that helps to determine the number of distinct classes of period $2n$ (where n is $x + y$). This equation works out much nicer when orbits that are not primitive are considered. Baxter and Umble then worked with this equation to find one that worked only with primitive orbits. To find out more about this equation and process, view Baxter and Umble's full paper [1].

This paper and the paper it is based on have brought up many points about periodic orbits on a mathematical billiards table. It would be interesting to see if any of these translated to a physical billiards table or if the spin of the ball completely changes the game. The unfolding technique to examine the orbit is a really cool idea. The pictures help to clarify how the technique works and it makes it much more concrete and easier to understand. The pictures that were created from unfoldings are actually what drew me into this topic. I would be interested to learn more about unfoldings on polygons other than equilateral triangles and see what applications, if any, those could have.

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