

Senior Seminar Proposal

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December 11, 2006

1 Introduction

A **Möbius transformation** of the complex plane is a linear fractional function $f : \mathbb{C} \rightarrow \mathbb{C}$ of the form

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are complex constants. (We allow $c = 0$, so that all linear transformations are Möbius). Möbius transformations have applications in physics and links to non-Euclidean geometries. They are examples of **conformal mappings**; that is, they preserve directed angles in the plane. They also map circles to circles.¹

Inversion in a circle is the analogue of reflection across a line.² To invert the point z in a circle centered at q and with radius R , find the point \tilde{z} in the same direction from q as z , but at a distance $\frac{R^2}{|q-z|}$. We then say that z and \tilde{z} are **symmetric** with respect to K . Inversion is useful for many reasons, not least its applications to non-Euclidean geometries. Primarily of interest here is its mapping of circles to circles.³ Specifically, if C is a circle passing through q , then its inverse \tilde{C} is a line parallel to the tangent at q . If C does not pass through q , then \tilde{C} is another circle not passing through q . Finally, if C is orthogonal to K , then $C = \tilde{C}$. See Figure 1.

Study up until now has focused on Möbius transformations of the complex plane. To this end, I've read through the pertinent sections of Tristan Needham's *Visual Complex Analysis*. (Namely, chapter three: "Möbius Transformations and Inversion").

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¹We allow that a line is a circle passing through ∞ .

²Indeed, given the above footnote, it is the generalization.

³It does *not*, however, map corresponding centers. If z is the center of C , then \tilde{z} is not the center of \tilde{C} .

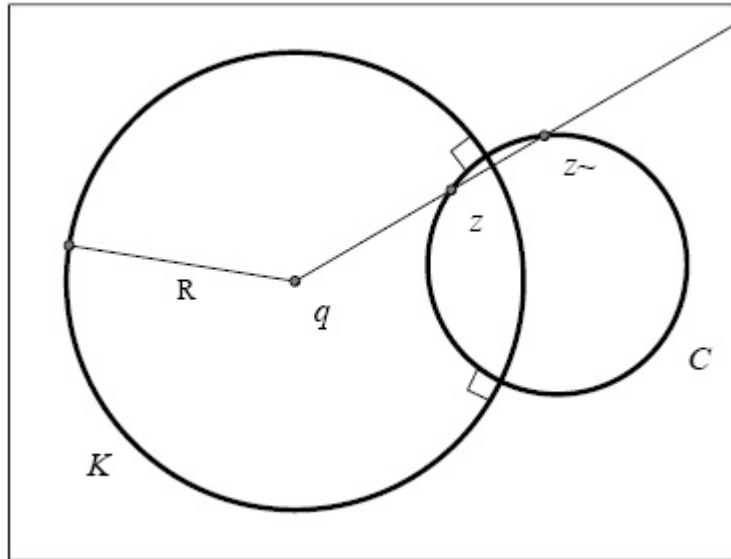


Figure 1: Inverting z in the circle K .

To begin, Needham notes that the Möbius transformation $f(z) = \frac{az+b}{cz+d}$ ($c \neq 0$) may be expressed as the composition of four simpler functions:

1. $z \mapsto z + \frac{d}{c}$ (translation)
2. $z \mapsto \frac{1}{z}$ (complex inversion)
3. $z \mapsto -z \frac{ad-bc}{c^2}$ (expansion and rotation)
4. $z \mapsto z + \frac{a}{c}$ (translation)

Moreover, complex inversion can be viewed as inversion in the unit circle followed by complex conjugation. To demonstrate that Möbius transformations preserve certain structures (circles and angles, in particular), Needham shows that they are preserved by each of the four simpler transformations.⁴ That translation, expansion, and rotation preserve these structures is rather trivial, but complex inversion requires a bit more thought. To that end, he discusses inversion in a circle at some length.

2 Work so far

In the course of doing so, he introduces a few geometric problems easily solved by inversion. One of these is his “touching circles” problem, in which

⁴When $c = 0$, we have a linear transformation.

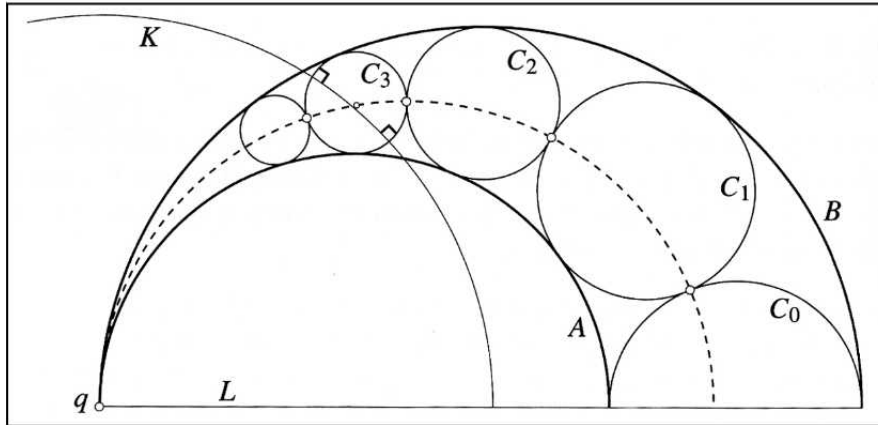


Figure 2: The touching circles problem. (Image taken from [1], p. 136).

two circles A and B (A interior to B) touch at a single point q . If we pack circles C_n between them as shown, so that C_n touches A , B , C_{n-1} , and C_{n+1} at a single point each, we find that their points of intersection lie on a circle. See Figure 2.

A major component of this project is to investigate what happens with the centers of the C_n . If we imagine moving C_0 around A so that it still touches A and B at a single point each, we see that its center traces out a curve C . We would like to know what sort of curve C is.

At first, it appeared that Needham's argument would also show that the centers lie on a circle. However, this was based on the faulty assumption that when inversion maps one circle to another, it also maps their corresponding centers.

It now appears that C is not a circle at all. For if C forms a circle C' , then it passes through both q and the center of C_0 , with center midway between them (by symmetry, the center of C' must be on the line containing q and the centers of A and B). When we draw this figure, however, we find (empirically) that any circle centered on C' and touching A intersects B twice or not at all.⁵ Thus, C is probably not a circle. See Figure 3.

3 Plans

Prior to realizing this error, I investigated a number of variations of this problem. Once I know what C is, I would like to give these situations more

⁵Except for C_0 , of course.

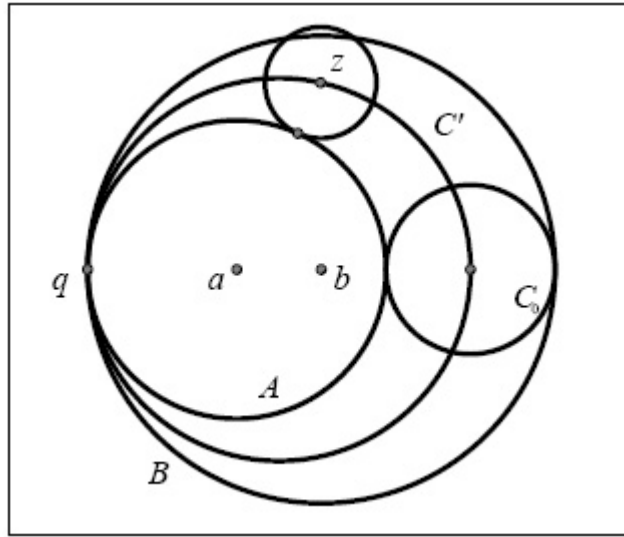


Figure 3: The circle centered at z touching A once intersects B twice.

thought. For instance, it is natural to ask what happens when A and B no longer touch, or else intersect at two points. Will the curve formed by the centers of the circles touching each at a single point be of the same form as C ? What sorts of defining characteristics will it have? When I thought C was a circle, it was natural to consider its center and radius. Depending on the nature of the curve, there may be similar questions to consider. (If it is an ellipse, for instance, we can ask about its foci and major and minor axes).

Another way to generalize this problem is to consider its analogue in three or higher dimensions. If we have spheres A and B in \mathbb{R}^3 , with A interior to B and touching at a single point, what sort of surface is formed by the centers of the spheres packed in between them? Is it related to the curve C in two dimensions? What happens when A and B no longer touch? What about in \mathbb{R}^n ? I hope to address all these questions in due time.

In addition to the touching circles problem, I hope to continue studying Möbius transformations. Möbius transformations in \mathbb{C} have analogues in \mathbb{R}^n , which are treated in Beardon's *Geometry of Discrete Groups*. Concurrently with research into the touching circles problem (or instead of it, in the case that I can't determine the nature of C), I hope to work through this text and familiarize myself with these analogues. Following that, I will either summarize results using other references, or look into any ideas I have along the way.

References

- [1] Needham, Tristan, *Visual Complex Analysis*, Clarendon Press, Oxford (1997).
- [2] Greenberg, Marvin J., *Euclidean and Non-Euclidean Geometries: Development and History*, W. H. Freeman and Company, New York (1997).
- [3] Beardon, Alan F., *The Geometry of Discrete Groups*, Springer-Verlag, New York (1983).