Möbius Transformations

Joseph W. Iverson*

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Abstract

In this paper, we study geometric inversions and the compositions thereof. We are already familiar with one kind of inversion in a familiar setting: reflection across a line in the plane. We'll see how this concept generalizes to inversion in a circle, and how these concepts in turn generalize to inversions in $\mathbb{R}^n$. Then, we'll define *Möbius transformations* as the finite compositions of these inversions and investigate some of their properties and applications.

1 Introduction

Inverses are a common theme in mathematics. Non-zero reals have reciprocals, group elements have their inverses, and if we're lucky, the function of the day is invertible.

Inverses needn't be related to groups, however. Commonly, we reflect entire two-dimensional figures across a given line, and this is a type of inversion. We can also invert in a circle by moving points far away in the exterior to points close to the center in the interior. These are examples of geometric inversions, and they are of particular interest here.

If we imagine inverting in a circle, then reflecting across a line, and finally inverting in some other circle, we see that the plane becomes very convoluted indeed. However, the transformation itself (called a *Möbius transformation*) is actually quite well-behaved. For instance, performing all the inversions in reverse order puts everything back where we started, so our transformation is a bijection. In fact, it is a homeomorphism. Moreover, if we track a circle

*Advisor: Xiaosheng Li

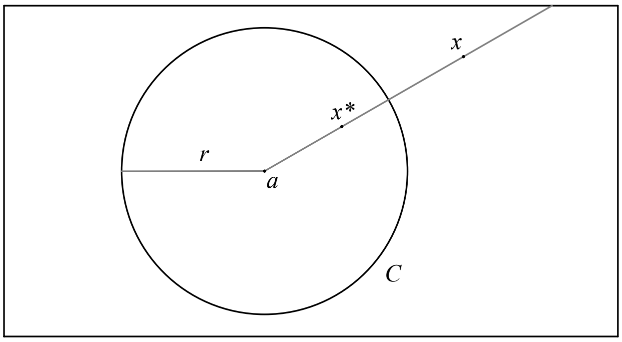

Figure 1: Inversion in a circle.

as its subjected to all these weird inversions, we find that the final outcome is either another circle or else a line. Even better, if we start with two circles intersecting at an angle θ, the resulting circles/lines will again meet at an angle of θ. These are just a few examples of the many elegant properties of Möbius transformations.

As always, we'd like to know if similar definitions apply in n-dimensional space, and see if we can derive similar results there. This generalization comprises the bulk of this paper. We rarely study a subject without its having applications to other areas of mathematics, so as an example of the usefulness of Möbius transformations, well talk about how to view stereographic projection as inversion in a circle, and apply that to a familiar example from topology. To begin, we'll give a more formal definition of inversion in a circle and briefly discuss its similarities with reflection in a line.

Motivating example: inverting in a circle

Definition 1 *If C is the circle with radius r and center a, and $x \neq a$ is any other point in the plane, then the* ***inverse*** *of x in C is*

$$x^* = a + \frac{r^2}{|x-a|^2}(x-a).$$

We see that x^* is on the ray $\overrightarrow{ax}$, at a distance $\frac{r^2}{|x-a|}$ from a, and also that x is the inverse of x^*. By virtue of the latter result, we say that inversion in a circle is **involutory**. To solve the problem of what happens to a under inversion in C, we attach another point to the plane. The resulting set is labeled $\hat{\mathbb{R}}^2 = \mathbb{R}^2 \cup \{\infty\}$, and we then define $a^* = \infty$.

As an alternative construction, we could construct two circles orthogonal to C and passing through x; their second intersection would then be x^*. If

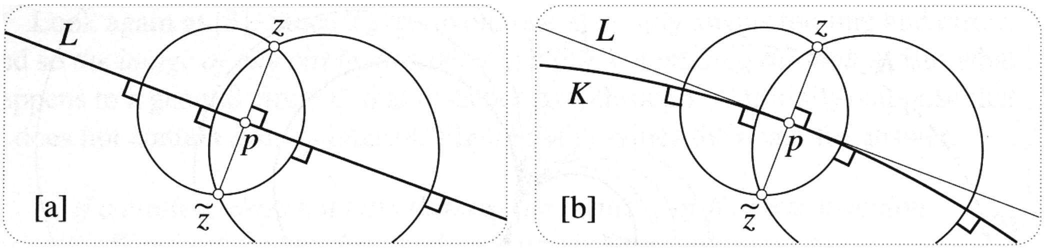

Figure 2: Inverting in a circle is like reflecting across a line. From [7], p. 130.

C were a line rather than a circle, this construction would give the reflection of x across it. In this regard, inversion in a circle is the generalization of reflection in a line (see Figure 2). Lines and circles often behave similarly in the realm of inversion, and for this reason we often think of lines as circles passing through the point ∞.

2 Inversions in $\mathbb{R}^n$

As is common in mathematics, we'd like to generalize the above case in two dimensions to one in n dimensions. For those unfamiliar with geometry in $\mathbb{R}^n$, we first establish a few preliminaries.

2.1 Preliminaries

As above, we'll find it useful to add another point to the plane, working in $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$.[1]

The **length** of a vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and it's easily shown that $d(x, y) = |x - y|$ is a metric (called the **Euclidean metric**).[2]

Just as in other spaces, a **sphere** is the set of points equidistant from its *center*. We denote

$$S(a, r) = \{x \in \mathbb{R}^n : |x - a| = r\},$$

[1] Later, we'll define a topology on this space that makes it the one-point compactification of $\mathbb{R}^n$.

[2] We'll find a metric on $\hat{\mathbb{R}}$ later. For now, sit tight and don't worry about the distance to ∞ or the topology of $\hat{\mathbb{R}}$.

and call r the *radius* (as you'd expect).

Dot products in n-dimensions are just like those in three dimensions; if $x = (x_1, \ \ldots, \ x_n)$ and $y = (y_1, \ \ldots, \ y_n)$, then the dot product of x and y is

$$(x \cdot y) = x_1 y_1 + \cdots + x_n y_n.$$

The following theorem gives a number of useful properties of the dot product; they are all quite easily verified:

Theorem 2 *For all $x, y, z \in \mathbb{R}^n$, and all $\lambda, \mu \in \mathbb{R}$:*

1. $(x \cdot y) = (y \cdot x)$
2. $(x \cdot x) = |x|^2$
3. $(x + y \cdot z) = (x \cdot z) + (y \cdot z)$
4. $(\lambda x \cdot y) = \lambda (x \cdot y)$
5. $|\lambda x + \mu y|^2 = |\lambda x|^2 + 2\lambda\mu(x \cdot y) + |\mu y|^2$.

As a particular consequence of the last point,

$$|x - y|^2 = |x|^2 - 2(x \cdot y) + |y|^2,$$

a fact that often proves useful when dealing with spheres.

Hyperplanes (or just *planes*) are the solutions to linear equations. Since each such equation can be written

$$(x \cdot a) = t$$

for some $a \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we denote planes by

$$P(a, t) = \{x \in \hat{\mathbb{R}}^n \colon (x \cdot a) = t \text{ or } x = \infty\}$$

(note that by convention each plane contains the point ∞).

Each $n \times n$ matrix A defines a linear transformation of $\mathbb{R}^n$, and we're particularly interested in those that preserve lengths (that is, those that satisfy $|Ax| = |x|$ for all $x \in \mathbb{R}^n$). These matrices are called **orthogonal**.

We're also interested in functions that preserve distance as calculated by the Euclidean metric. We call these functions **Euclidean isometries**.

Finally, we'll run into a group of invertible matrices at least once. In that instance, we'll use the standard notation

$$GL_n(F) = \{\text{invertible } n \times n \text{ matrices with entries in the field } F\},$$

where group multiplication is given by matrix multiplication.

2.2 Reflections in spheres

Obviously, the right way to generalize inversion in circle in $\mathbb{R}^2$ to $\mathbb{R}^n$ is to define inversion in a sphere. In two dimensions, the inverse of x in $C(a,r)$ cut the ray $\overrightarrow{ax}$ to length $\frac{r^2}{|x-a|}$. The equivalent of $\overrightarrow{ax}$ in $\mathbb{R}^n$ is the set $\{a+\lambda(x-a) \colon \lambda > 0\}$. The parameter λ controls distance from a;

$$|\,(a+\lambda(x-a)) - a| = \lambda|x-a|.$$

We're looking for the point at a distance $\frac{r^2}{|x-a|}$ from a, so we set

$$\begin{aligned} \lambda|x-a| &= \frac{r^2}{|x-a|} \\ \lambda &= \frac{r^2}{|x-a|^2}. \end{aligned}$$

We also want a and ∞ to be inverses. Lacking a tidier way to incorporate this into our definition, we specify it.

Definition 3 *The* ***inverse*** *or* ***reflection*** *of x in $S(a,r)$ is given by*

$$\phi(x) = \begin{cases} a & \text{if } x = \infty; \\ \infty & \text{if } x = a; \\ a + \left(\frac{r}{|x-a|}\right)^2 (x-a) & \text{otherwise.} \end{cases}$$

Inversion in the unit sphere $S^{n-1} = S(0,1)$ is a special case that comes up often[3], so have a special notation for it:

$$x^* = \frac{x}{|x|^2}.$$

This gives the equivalent formulation

$$\phi(x) = a + r^2(x-a)^*, \tag{1}$$

where infinity is treated as you'd expect.

By composing ϕ with itself in the above formula, you can easily see that inversion in a sphere is involutory. It is therefore its own inverse, and must be a bijection.

[3]As an example, the inverse of $x \in \mathbb{R}$ in S^0 is x^{-1}.

We can also use (1) to find the distance between inverse points:

$$\begin{aligned}
|\phi(y)-\phi(x)| &= |a+r^2(y-a)^* - a - r^2(x-a)^*| \\
&= r^2\left|\frac{(y-a)}{|y-a|^2} - \frac{(x-a)}{|x-a|^2}\right| \\
&= r^2\sqrt{\frac{1}{|y-a|^2} - \frac{2(x-a\cdot y-a)}{|x-a|^2|y-a|^2} + \frac{1}{|x-a|^2}} \\
&= r^2\sqrt{\frac{|(x-a)-(y-a)|^2}{|x-a|^2|y-a|^2}} \\
&= \frac{r^2|x-y|}{|x-a||y-a|},
\end{aligned}$$

when $x, y \neq \infty$, and (similarly)

$$|\phi(y)-\phi(\infty)| = \frac{r^2}{|y-a|}.$$

One other result merits immediate attention:

Theorem 4 *If Σ is any plane or sphere and ϕ is inversion in $S(a,r)$, then $\phi(\Sigma)$ is a plane when $a \in \Sigma$ and a sphere otherwise.*

The specific form of $\phi(\Sigma)$ is derived in an appendix, and we take that as proof.

2.3 Reflections across planes

To be complete, we also generalize reflection across a line. Lines are just the solutions to linear equations, so we generalize to $\hat{\mathbb{R}}^n$ with reflection across a plane. To see how, think of the case in two dimensions. The reflection of a point x across a line ℓ is another point x', chosen so that $\overline{xx'}$ is perpendicular to ℓ, with midpoint on the line.

The analogue of $\overleftrightarrow{xx'}$ in $\mathbb{R}^n$ is the line

$$\{x+\lambda(x-x') \colon \lambda \in \mathbb{R}\},$$

whose "direction" is given by the vector $x - x'$. Just as in three dimensions, the normal vector to the plane

$$P(a,t) = \{y \colon y_1a_1 + y_2a_2 + \cdots + y_na_n = t \text{ or } y = \infty\}$$

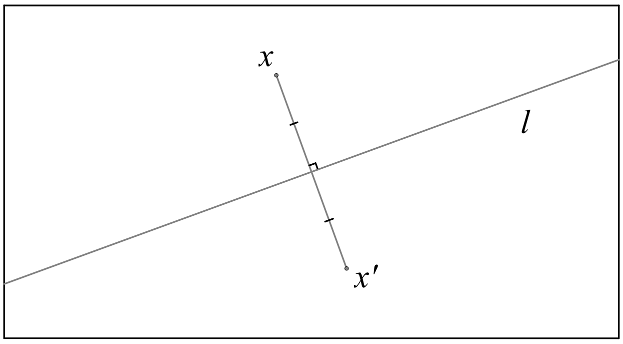

Figure 3: Reflection across a line.

is $(a_1, a_2, \ldots, a_n) = a$, so x's inverse x' should be chosen so that $x - x'$ is a scalar product of a:

$$x - x' = \lambda_x a.$$

To find λ_x, we ensure that the "midpoint" of $\overline{xx'}$ is on $P(a,t)$:

$$\begin{aligned}
\left(\frac{x+x'}{2} \cdot a\right) &= t \\
\left(x - \frac{1}{2}(x - x') \cdot a\right) &= t \\
\left(x - \frac{1}{2}\lambda_x a \cdot a\right) &= t \\
(x \cdot a) - \frac{1}{2}\lambda_x |a|^2 &= t \\
\frac{2\left[(x \cdot a) - t\right]}{|a|^2} &= \lambda_x.
\end{aligned}$$

This gives us the reflection of all finite points. Lacking a suitable alternative, we reflect ∞ upon itself.

Definition 5 *The **inverse** or **reflection** of a point x across the plane $P(a,t)$ is given by*

$$\phi(x) = \begin{cases} x - 2\left[(x \cdot a) - t\right] a^*, & \text{if } x \in \mathbb{R}^n; \\ \infty, & \text{if } x = \infty. \end{cases}$$

Just like inversion in a sphere, inversion in a plane is involutory, and therefore bijective.

As an example, consider reflection across the plane $x_n = k$ (that is, across $P(e_i, k)$, where e_i is the unit vector on the i^{th} axis). Since $e_i \in S^{n-1}$, $e_i^* = e_i$. If $x = (x_1, \ldots, x_i, \ldots, x_n)$, then

$$\begin{aligned} \phi(x) &= x - 2(x_i - k)e_i \\ &= (x_1, \ldots, 2k - x_i, \ldots, x_n). \end{aligned}$$

Theorem 6 *Reflection across a plane is a Euclidean isometry.*

Proof. Let $\phi(x)$ be inversion in $P(a,t)$. If we define

$$\lambda_x = \frac{-2\,[(x \cdot a) - t]}{|a|^2},$$

then

$$\begin{aligned} |\phi(y) - \phi(x)|^2 &= |(y + \lambda_y a) - (x + \lambda_x a)|^2 \\ &= |(y - x) + (\lambda_y - \lambda_x)a|^2 \\ &= |y - x|^2 + 2(\lambda_y - \lambda_x)\,[(y \cdot a) - (x \cdot a)] + (\lambda_y - \lambda_x)^2 |a|^2. \end{aligned}$$

Since $(\lambda_y - \lambda_x) = \frac{-2}{|a|^2}\,[(y \cdot a) - (x \cdot a)]$, we have

$$|\phi(y) - \phi(x)|^2 = |y - x|^2,$$

and the result follows.

The following will therefore prove useful; its proof appears in [1].

Theorem 7 *Each Euclidean isometry $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ can be written*

$$\phi(x) = Ax + \phi(0),$$

where A is some orthogonal matrix and x is written as a column vector.

We also have the following analogue to Theorem 4:

Theorem 8 *If ϕ is reflection in a plane and Σ is any plane or sphere, then $\phi(\Sigma)$ is another plane or sphere, respectively.*

As before, the specific form of $\phi(\Sigma)$ appears in an appendix, which serves as sufficient proof.

As another example, observe that reflection ϕ in a plane $P(a,t)$ translates both the unit sphere and the **unit ball**

$$B^n = \{x \colon |x| < 1\}$$

by a factor of $\phi(0)$. Thus, they remain invariant if and only if $t = 0$.

2.4 Möbius transformations

We are now ready to discuss the main topic of this paper: Möbius transformations and the properties thereof.

Definition 9 *A **Möbius transformation** is the finite composition of inversions in planes and spheres.*

We immediately observe that the Möbius transformations on $\hat{\mathbb{R}}^n$ form a group under composition; it is the group generated by all reflections in planes or spheres. Obviously, the composition of Möbius transformations must be Möbius, and since each inversion has an inverse, so does each Möbius transformation. Specifically, if $\phi = \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n$, then $\phi^{-1} = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$. The identity mapping is also Möbius; it is the composition of zero inversions. We call the group of Möbius transformation on $\hat{\mathbb{R}}^n$ the **general Möbius group** and denote it $GM(\hat{\mathbb{R}}^n)$. The set of compositions of an even number of reflections is a subgroup, called the **Möbius group** and denoted $M(\hat{\mathbb{R}}^n)$.

We have one more immediate result, which follows from Theorems 4 and 8.

Corollary 10 *If ϕ is any Möbius transfromation and Σ is any plane or sphere, then $\phi(\Sigma)$ is a plane or a sphere.*

The set of Möbius transformations includes many more functions than that of inversions alone.[4] For instance, each expansion $x \mapsto kx$ $(k > 0)$ is inversion in the unit sphere followed by inversion in $S(0, \sqrt{k})$, as can easily be checked. Similarly, the translation $x \mapsto x + a$ is inversion in $P(a, 0)$ followed by that in $P(a, \frac{1}{2}|a|^2)$.[5] In fact, every Euclidean isometry is the composition of reflections in planes (and therefore Möbius).

In the extended complex plane $\hat{\mathbb{C}} = \hat{\mathbb{R}}^2$ (with complex multiplication), we can further specify that a transformation $f : \mathbb{C} \to \mathbb{C}$ is in $M(\hat{\mathbb{R}}^2)$ iff it has the form

$$f(z) = \frac{az + b}{cz + d},$$

[4] How many more? To answer this, we turn to dimension theory. The space of inversions has dimension $(n+1)$, while the space of Möbius transformations has dimension $\frac{(n+1)(n+3)}{2}$. In rough terms, the difference between these spaces is like the difference between $\mathbb{R}^{n+1}$ and $\mathbb{R}^{\frac{(n+1)(n+3)}{2}}$.

[5] These results both come directly from [1], p. 23. Not every linear transformation of $\mathbb{R}^n$ is Möbius; the projection $\pi_i : (x_1, \ldots, x_n) \mapsto (0, \ldots, x_i, \ldots, 0)$ lacks an inverse.

where $a, b, c, d \in \mathbb{C}$ are constants. In fact, if we denote

$$[f] = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

we find that

$$[f]\,[g] = [f \circ g]\,,$$

as can easily be verified. The mapping $\rho\colon [f] \mapsto f$ is thus a homomorphism from $GL_2(\mathbb{C})$ to $M(\hat{\mathbb{R}}^2)$.[6]

3 Diversion: a little topology

We're coming up on a few results using or related to basic topology. For readers unfamiliar with the subject, we offer this brief introduction.

The **topology** of a space determines which functions leaving and entering the space are continuous. Formally, it is a collection of subsets of the space (deemed the "open sets") satisfying three axioms:

1. The empty set and the whole space are open;

2. The intersection of any two open sets is open;

3. The union of any collection of open sets is open.

Continuous functions then satisfy a condition familiar from analysis; namely, $f\,:\; X \to Y$ is continuous iff $f^{-1}(V)$ is open for each open set $V \subset Y$.

A given space can have many different topologies. In that regard, topologies are sort of like neckties: on any given day, you can choose to use a really goofy topology, a really boring topology, or a topology like you see all the time. Just as with neckties, it's important to pick the right one.

Each metric on a space X defines a topology by taking as open sets all unions of finite intersections of balls $B_d(x, r) := \{y \in X : d(x, y) < r\}$. There is no discrepancy between functions continuous with respect to this topology and functions continuous by the metric definition. Other terminology from analysis carries over just as naturally; a **compact space** is still one where every covering by open sets has a finite sub-covering, for instance.

[6] We'd like to have it the other way around, from $M(\hat{\mathbb{R}}^2)$ to $GL_2(\mathbb{C})$. Such a mapping from a group G to $GL_n(F)$ is called a **representation**, and is very useful for determining properties of the group. Unfortunately, multiplying $[f]$ by any non-zero constant gives the same transformation f. That is, each function $f \in M(\hat{\mathbb{R}}^2)$ is associated with a whole slew of matrices in $GL_2(\mathbb{C})$. Normalization can cut this number down, but the relation $f \sim [f]$ is still one-to-two, as $f \sim -[f]$.

In analysis, we often talk about the *completion* of a metric space M: the complete metric space containing M as a dense subspace. Just so, topologists often discuss the **compactifications** of a topological space X: those compact spaces Y with X as a dense[7] topological subspace. One major result from topology is that every topological space can be compactified by adding a single point to it, and if the space is nice enough[8], this **one-point compactification** is unique up to homeomorphism (still a bi-continuous bijection).

4 Stereographic projection and $\hat{\mathbb{R}}^n$

To solve the problem of what happens to the center of a sphere under inversion, we've unceremoniously stuck another point ∞ onto Euclidean space. While this appropriately signifies that points closer to the center of the sphere are projected further away by inversion, we have yet to account for how ∞ fits into the topology of $\hat{\mathbb{R}}^n$. Without an appropriate image of the space, it is difficult to tell which open sets ought to include the point.

In short, the solution is to project the entirety of $\hat{\mathbb{R}}^n$ onto a sphere in $\hat{\mathbb{R}}^{n+1}$. Using distance by chords, we'll derive a topology identical to the standard one[9] when restricted to $\mathbb{R}^n$, but which now includes the point ∞ in an appropriate way. In doing so, we'll show that $\hat{\mathbb{R}}^n$ is actually the one-point compactification of $\mathbb{R}^n$.

4.1 Sample case: $\hat{\mathbb{R}} \mapsto S^1$

As an example, we'll first project $\hat{\mathbb{R}}$ onto the unit circle in $\hat{\mathbb{R}}^2$. For each $x \in \mathbb{R}$, draw the line joining $(x, 0)$ with $(0, 1)$ and find its second intersection with S^1. Label this point $\tilde{x}$. The function

$$\pi \colon x \mapsto \begin{cases} \tilde{x} & \text{if } x \in \mathbb{R}; \\ (0,1) & \text{if } x = \infty. \end{cases}$$

is called the **stereographic projection**[10] of $\hat{\mathbb{R}}$ onto S^1.

[7] We won't go into detail about precisely what this means in a topological space, but suffice it to say that it is very much analogous to density in a metric space.

[8] Specifically, this holds when the space is *locally compact*; see [2], p. 246. $\mathbb{R}^n$ is locally compact.

[9] That is, the topology induced by the Euclidean metric.

[10] Strangely, stereographic projections aren't what we'd normally call projections. We can't compose π with itself at all, much less have $\pi^2 = \pi$.

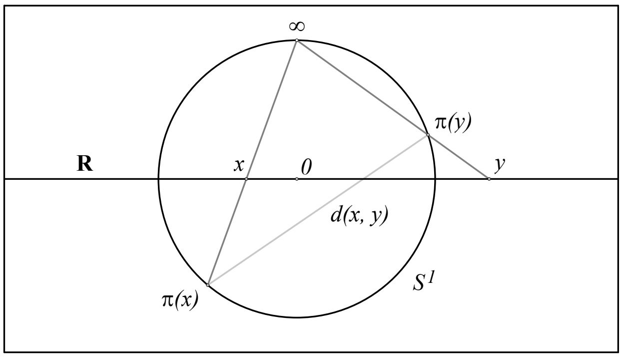

Figure 4: Stereographic projection.

It's easy to see that $\pi\colon \hat{\mathbb{R}} \to S^1$ is injective, from which it follows that

$$d(x, y) = |\pi(x) - \pi(y)|$$

is a metric (called the **chordal metric**) on $\hat{\mathbb{R}}$:

$$\begin{aligned} |\pi(x) - \pi(y)| = 0 \quad &\text{iff} \quad \pi(x) = \pi(y) \\ &\text{iff} \quad x = y \end{aligned}$$

(the other properties flow down from the Euclidean metric).

4.2 $\hat{\mathbb{R}}^n \to S^n$

We now generalize this procedure to n dimensions. To formalize it, we use reflection in a sphere. To begin, identify $\hat{\mathbb{R}}^n$ with the plane $P(e_{n+1}, 0) \subset \hat{\mathbb{R}}^{n+1}$ via the embedding

$$x \mapsto \bar{x} = \begin{cases} (x_1, x_2, \ldots, x_n, 0), & \text{if } x = (x_1, x_2, \ldots, x_n); \\ \infty, & \text{if } x = \infty. \end{cases}$$

Inverting in the sphere $S(e_{n+1}, \sqrt{2})$, $P(e_{n+1}, 0)$ becomes S^n.[11] Informally, this inversion sends each point $x \in P(e_{n+1}, 0) - \{\infty\}$ to another point on the line joining x and e_{n+1}; it is the point on S^n.

Definition 11 *The* ***stereographic projection*** *of $\hat{\mathbb{R}}^n$ onto S^n is given by the function*

$$\pi\colon \hat{\mathbb{R}}^n \to S^n, \;\; x \mapsto \phi(\bar{x}),$$

where ϕ is inversion in $S(e_{n+1}, \sqrt{2})$ and $\bar{x}$ is as above.

[11] The general form of a plane reflected in a sphere is given in an appendix. We use that result here.

As before, we have the chordal metric $d(x, y) = |\pi(x) - \pi(y)|$. More explicitly,

$$d(x,y) = \begin{cases} \frac{2|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} & \text{if } x, y \neq \infty; \\ \frac{2}{\sqrt{1+|x|^2}} & \text{if } y = \infty. \end{cases}$$

This solves the problem of which topology to use on $\hat{\mathbb{R}}^n$. The one induced by d is the same as the Euclidean metric when restricted to $\mathbb{R}^n$. Hence, a transformation $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous with respect to one metric if and only if it is continuous with respect to the other.

We can easily see that reflection in $S(a, r)$ is continuous for all $x \in \mathbb{R}^n - \{a\}$. To establish continuity on all of $\hat{\mathbb{R}}^n$, note that

$$\begin{aligned} \lim_{x \to a} d(\phi(x), \phi(a)) &= \lim_{x \to a} d(\phi(x), \infty) \\ &= \lim_{x \to a} \frac{2}{\sqrt{1 + |\phi(x)|^2}} \\ &= 0, \end{aligned}$$

and likewise

$$\begin{aligned} \lim_{x \to \infty} d(\phi(x), \phi(\infty)) &= \lim_{x \to \infty} d(\phi(x), a) \\ &= \lim_{x \to \infty} \frac{2|\phi(x) - a|}{\sqrt{1 + |\phi(x)|^2}\sqrt{1 + |a|^2}} \\ &= 0. \end{aligned}$$

Reflections across planes are also continuous, and we therefore have:

Theorem 12 *Each Möbius transformation $\phi\colon \hat{\mathbb{R}}^n \to \hat{\mathbb{R}}^n$ is a homeomorphism.*

We now see that stereographic projection is a homeomorphism, and as S^n is compact, so is $\hat{\mathbb{R}}^n$. Hence,

Corollary 13 *$\hat{\mathbb{R}}^n$ is the one-point compactification of $\mathbb{R}^n$, and every one-point compactification of $\mathbb{R}^n$ is homeomorphic to S^n.*

5 Möbius invariants

As with any function, we're keen to know what sorts of things remain unchanged by Möbius transformations. We've already seen that the class of all spheres and planes is left alone. As it turns out, a few more things related to this class remain invariant.

In two dimensions, we often think of lines as circles passing through infinity. In fact, with stereographic projection this is literally true.[12] We adopt a similar convention in this section, using the word "sphere" to refer to both Euclidean spheres of the form $S(a, r)$ and planes of the form $P(a, t)$. We'll take pains to distinguish between these when necessary.

5.1 Inversive products and conformality

Inversion in a circle preserves the angle between tangent lines to intersecting curves. For this reason, we say that it is **conformal**. In particular, the angle between any two circles is the same as that between their images.

In $\hat{\mathbb{R}}^n$, we have a similar result. We can write any sphere $S(a, r)$ as the set of all points satisfying

$$|x|^2 - 2(x \cdot a) + \left(|a|^2 - r^2\right) = 0,$$

and any plane $P(a, t)$ as those points satisfying

$$-2(x \cdot a) + 2t = 0.$$

In general, any sphere is the set of solutions to some equation

$$\lambda|x|^2 - 2(x \cdot a) + \mu = 0.$$

The **coefficient vector** of that sphere is then defined as

$$(\lambda, a_1, a_2, \ldots, a_n, \mu) \in \mathbb{R}^{n+2}.$$

As you can easily see, it is unique up to a scalar multiple.

We can use these vectors to find a unique number indicative of the angle between any two spheres.

Definition 14 *Let* $f \colon \mathbb{R}^{n+2} \times \mathbb{R}^{n+2} \to \mathbb{R}$ *be given by*

$$f(x, y) = 2(x_2 y_2 + x_3 y_3 + \cdots + x_{n+1} y_{n+1}) - (x_1 y_{n+2} + x_{n+2} y_1).$$

If Σ *and* Σ' *are spheres with coefficient vectors* a *and* b*, respectively, then their* ***inversive product*** *is*

$$(\Sigma, \Sigma') = \frac{|f(a, b)|}{\sqrt{|f(a, a)||f(b, b)|}}.$$

[12] Does this hold in general? Suppose we are given a plane $P \subset \hat{\mathbb{R}}^n$. Find its stereographic $\pi(P)$ in S^n, and see if there is a plane $P' \subset \hat{\mathbb{R}}^{n+1}$ containing P. By translating and rotating P', we can line it up with $P(e_{n+1}, 0)$ and therefore identify it with $\hat{\mathbb{R}}^n$. Is the image of $\pi(P)$ under this transformation a sphere in $\hat{\mathbb{R}}^n$?

The inversive product indicates the angle of intersection: $\cos\theta = (\Sigma, \Sigma')$. In particular, two spheres are called **orthogonal** iff $(\Sigma, \Sigma') = 0$. As in two dimensions, this angle is preserved.

Theorem 15 *Möbius transformations preserve inversive products.*

In fact, a much more general result holds. As in two dimensions, we can find the tangent vector to a given curve in $\hat{\mathbb{R}}^n$, and we can find the angle θ between vectors u and v by defining

$$(u \cdot v) = |u|\,|v| \cos\theta.$$

The angle given by the inversive product is just a special case of this. Each reflection preserves this angle but reverses its orientation.

Theorem 16 *Each Möbius transformation is conformal. Those that are compositions of an odd number of reflections reverse orientation, while those that are compositions of an even number preserve it.*

5.2 The inverse relation

Inversion in a sphere maps that sphere to itself. In fact, basically no other Möbius transformation has that property.

Theorem 17 *If Σ is a sphere invariant under a Möbius transformation ϕ, then ϕ is either inversion in Σ or the identity map.*

We mention this theorem to prove another, which tells us that the relation of being an inverse in a sphere is more or less preserved.

Theorem 18 *If a and b are inverses in a sphere Σ and ϕ is any Möbius transformation, then $\phi(a)$ and $\phi(b)$ are inverses in $\phi(\Sigma)$.*

Proof. Let σ be reflection in Σ, and let ρ be reflection $\phi\,(\Sigma)$. Since ρ keeps $\phi(\Sigma)$ constant, $\rho\phi$ maps $\Sigma \mapsto \phi(\Sigma)$. Hence, the composition $\phi^{-1}\rho\sigma$ holds Σ invariant. By the above, it is either reflection in Σ or the identity mapping. In the later case,

$$\phi^{-1}\rho\phi = \iota \implies \rho = \iota,$$

which is clearly false. Hence, $\rho\phi = \phi\sigma$. If $y = \sigma(x)$ is the inverse of x in Σ, then

$$\begin{aligned} \rho\,(\phi(x)) &= \phi\,(\sigma(x)) \\ &= \phi(y), \end{aligned}$$

so $\phi(x)$ and $\phi(y)$ are inverses in $\phi(\Sigma)$.

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Appendix: Images of planes and spheres under reflection

In two dimensions, it is known how to map any circle onto another using inversion. We'd like to generalize this result to n dimensions. That is, given spheres $S(a, r)$ and $S(b, s)$, we would like to find a sphere, reflection in which maps one to the other.

Spheres with equal radii

Anticipating an easy result, we first consider the case where $r = s$. In two dimensions, reflection across a line would work, so intuitively, reflection in a plane should suffice here. It's not clear which plane to use, so we'll try $P\left(\frac{1}{2}(a-b), t\right)$ and find t so that a and b are inverses. Let ϕ be inversion in $P\left(\frac{1}{2}(a-b), t\right)$. Then

$$\begin{aligned}\phi(a) &= a + \frac{\lambda}{2}(a-b) \\ &= \left(1 + \frac{\lambda}{2}\right) a - \frac{\lambda}{2} b,\end{aligned}$$

where

$$\lambda = \frac{-2\left[\left(a \cdot \frac{1}{2}(a-b)\right) - t\right]}{\left|\frac{1}{2}(a-b)\right|^2}.$$

To simplify this, note that

$$\begin{aligned}\left(a \cdot \frac{1}{2}(a-b)\right) &= \frac{1}{2}(a \cdot a) - \frac{1}{2}(a \cdot b) \\ &= \frac{1}{2}|a|^2 - \frac{1}{2}(a \cdot b),\end{aligned}$$

and

$$\left|\frac{1}{2}(a-b)\right|^2 = \frac{1}{4}|a|^2 - \frac{1}{2}(a \cdot b) + \frac{1}{4}|b|^2.$$

Hence,

$$\begin{aligned}\lambda &= \frac{-2\left[\frac{1}{2}|a|^2 - \frac{1}{2}(a \cdot b) - t\right]}{\frac{1}{4}|a|^2 - \frac{1}{2}(a \cdot b) + \frac{1}{4}|b|^2} \\ &= \frac{-4|a|^2 + 4(a \cdot b) + 8t}{|a|^2 - 2(a \cdot b) + |b|^2}.\end{aligned}$$

We want $\lambda = -2$, so we set

$$\begin{aligned} -4|a|^2 + 4(a \cdot b) + 8t &= -2|a|^2 + 4(a \cdot b) - 2|b|^2 \\ 8t &= 2|a|^2 - 2|b|^2 \\ t &= \frac{1}{4}|a|^2 - \frac{1}{4}|b|^2. \end{aligned}$$

Reflecting in a plane preserves Euclidean distance, so

$$|x - a| = r \iff |\phi(x) - b| = r.$$

Hence, $S(a, r)$ and $S(b, s)$ are inverses in the plane

$$P\left(\frac{1}{2}(a - b), \frac{1}{4}|a|^2 - \frac{1}{4}|b|^2\right).$$

Spheres reflected in spheres

To help this process (and because our interest is piqued), we look for the generic form of the inverse of a sphere. Observe that, due to (1), inversion in $\Sigma = S(a, r)$ consists of four steps taken in order:

1. Translate by $-a$;
2. Invert in the unit sphere;
3. Expand by r^2;
4. Translate by a.

It's pretty easy to see what the translation and expansion of a given sphere is, so we only need to find the generic form of a sphere $S(c, s)$ inverted in S^{n-1}.

As we've seen, each sphere can be viewed as the set of points satisfying some equation

$$\lambda|x|^2 - 2(x \cdot c) + \mu = 0.$$

If $y = x^*$, so that $|x| = \frac{1}{|y|}$ and $x = \frac{1}{|y|^2}y$, then the above equation becomes

$$\lambda - 2(y \cdot c) + |y|^2\mu = 0. \tag{2}$$

Points on $S(c, s)$ satisfy

$$|x|^2 - 2(x \cdot c) + |c|^2 - 2 = 0,$$

so in this case,

$$\mu = |c|^2 - s^2, \;\; \lambda = 1.$$

Hence, the inverse of $S(c,s)$ is the set of points y satisfying

$$1 - 2(y \cdot c) + \left(|c|^2 - s^2\right)|y|^2 = 0.$$

If $|c|^2 = s^2$ (i.e. $0 \in S(c,s)$), this is the plane $P(c, \frac{1}{2})$. Otherwise, we transform the above equation to

$$|y|^2 - \frac{2}{|c|^2 - s^2}(y \cdot c) + \frac{1}{|c|^2 - s^2} = 0. \tag{3}$$

As we'll see, this is a sphere, but it's center and radius are not yet apparent.

In general, suppose $y \in \mathbb{R}^n$ satisfies

$$|y|^2 - \alpha(y \cdot a) + \beta = 0. \tag{4}$$

Writing this out, we have

$$(y_1^2 + y_2^2 + \cdots + y_n^2) - (\alpha y_1 a_1 + \alpha y_2 a_2 + \cdots + \alpha y_n a_n) + \beta = 0.$$

By completing the square, we derive

$$(y_1 - \frac{\alpha}{2}a_1)^2 + (y_2 - \frac{\alpha}{2}a_2)^2 + \cdots + (y_n - \frac{\alpha}{2}a_n)^2 - \frac{\alpha^2}{4}(a_1^2 + a_2^2 + \cdots + a_n^2) + \beta = 0$$

$$\left|y - \frac{\alpha}{2}a\right|^2 = \frac{\alpha^2}{4}|a|^2 - \beta.$$

Hence, the set of y satisfying (4) is

$$S\left(\frac{\alpha}{2}a, \sqrt{\left(\frac{\alpha|a|}{2}\right)^2 - \beta}\right).$$

It's now clear that (3) gives a sphere. Applying the above with

$$\beta = \frac{1}{|c|^2 - s^2}, \;\; \alpha = \frac{2}{|c|^2 - s^2} = 2\beta,$$

we see it must be centered at

$$\begin{aligned} \frac{\alpha}{2}c &= \beta c \\ &= \left(\frac{1}{|c|^2 - s^2}\right)c, \end{aligned}$$

with radius

$$\begin{aligned}\sqrt{\frac{\alpha^2}{4}|c|^2-\beta} &= \sqrt{\beta^2|c|^2-\beta} \\ &= \sqrt{\frac{|c|^2}{(|c|^2-s^2)^2}-\frac{1}{|c|^2-s^2}} \\ &= \frac{s}{||c|^2-s^2|}.\end{aligned}$$

Suppose now that we are inverting the sphere $S(b,s)$ in $S(a,r)$. We'll keep track of its center and radius as we go through the four steps of inversion.

After Step 1:	Center:	$b-a$
	Radius:	s
After Step 2:	Center:	$\left(\frac{1}{|b-a|^2-s^2}\right)(b-a)$
	Radius:	$\frac{s}{||b-a|^2-s^2|}$
After Step 3:	Center:	$\left(\frac{r^2}{|b-a|^2-s^2}\right)(b-a)$
	Radius:	$\frac{r^2 s}{||b-a|^2-s^2|}$
After Step 4:	Center:	$a+\left(\frac{r^2}{|b-a|^2-s^2}\right)(b-a)$
	Radius:	$\frac{r^2 s}{||b-a|^2-s^2|}$

In step 2, we assume $|b-a|^2 \neq s^2$. Otherwise, we have a plane. To see which plane, we'll need some preliminary results. If $x \in P(a,t)-\{\infty\}$, then

$$\begin{aligned}(x+b\cdot a) &= (x\cdot a)+(b\cdot a) \\ &= t+(a\cdot b),\end{aligned}$$

and

$$\begin{aligned}(kx\cdot a) &= k(x\cdot a) \\ &= kt.\end{aligned}$$

Thus, translating $P(a,t)$ by b gives $P\left(a, t+(a\cdot b)\right)$, and expanding it by a factor of k gives $P(a,kt)$.

When $|b-a|^2 = s^2$ (i.e. $a \in S(b,s)$), inverting $S(b-a,s)$ in the unit sphere gives the plane $P\left(b-a,\frac{1}{2}\right)$. Expanding by a factor of r^2 then gives $P(b-a,\frac{r^2}{2})$, and translating by a produces the plane

$$P\left(b-a,\frac{r^2}{2}+(a\cdot b)-|a|^2\right)$$

as the inverse of $S(b,s)$ in $S(a,r)$.

In summary,

Theorem 19 *The reflection of $S(b,s)$ in $S(a,r)$ is*

$$P\left(b-a, \frac{r^2}{2}+(a\cdot b)-|a|^2\right)$$

when $a \in S(b,s)$, and

$$S\left(a+\left(\frac{r^2}{|b-a|^2-s^2}\right)(b-a), \frac{r^2 s}{||b-a|^2-s^2|}\right)$$

otherwise.

Spheres with distinct radii

Let's return to the problem of mapping any sphere $S(b,s)$ onto another sphere $S(c,t)$ via a Möbius transformation. When $s=t$, we've found that we can just invert in a plane. To solve the other case, we try inverting in some sphere $S(a,r)$ and assume it maps $S(b,s)$ to $S(c,t)$. Hence,

- $t=\frac{r^2 s}{||b-a|^2-s^2|} \implies r^2=\frac{t\left||b-a|^2-s^2\right|}{s}$, and
- $c=a+\left(\frac{r^2}{|b-a|^2-s^2}\right)(b-a)$.

Note that

$$\frac{r^2}{|b-a|^2-s^2}=\begin{cases} \frac{t}{s}, & \text{if } |b-a|^2-s^2>0 \\ -\frac{t}{s}, & \text{if } |b-a|^2-s^2<0. \end{cases}$$

We consider these cases separately. First assume $|b-a|^2>s^2$. Then

$$\begin{aligned} a+\frac{t}{s}(b-a) &= c \\ \left(1-\frac{t}{s}\right)a &= c-\left(\frac{t}{s}\right)b \\ a &= \left(\frac{s}{s-t}\right)c-\left(\frac{t}{s-t}\right)b. \end{aligned}$$

In this case,

$$\begin{aligned} |b-a|^2 &= \left|b-\left(\frac{s}{s-t}\right)c+\left(\frac{t}{s-t}\right)b\right|^2 \\ &= \left(\frac{s}{s-t}\right)^2|b-c|^2, \end{aligned}$$

which is greater than s^2 iff

$$|b-c| > |s-t|.$$

We also have

$$\begin{aligned} r &= \sqrt{\frac{t\left(|b-a|^2 - s^2\right)}{s}} \\ &= \sqrt{\left(\frac{ts}{(s-t)^2}\right)|b-c|^2 - ts}. \end{aligned}$$

Now assume $|b-a|^2 < s^2$. Then

$$\begin{aligned} a - \frac{t}{s}(b-a) &= c \\ \left(1+\frac{t}{s}\right)a &= c + \left(\frac{t}{s}\right)b \\ a &= \left(\frac{s}{s+t}\right)c + \left(\frac{t}{s+t}\right)b. \end{aligned}$$

In this case,

$$\begin{aligned} |b-a|^2 &= \left|b - \left(\frac{s}{s+t}\right)c - \left(\frac{t}{s+t}\right)b\right| \\ &= \left(\frac{s}{s+t}\right)^2 |b-c|^2, \end{aligned}$$

which is less than s^2 iff

$$|b-c| < s+t.$$

We also have

$$\begin{aligned} r &= \sqrt{\frac{t}{s}\left(s^2 - |b-a|^2\right)} \\ &= \sqrt{ts - \left(\frac{ts}{(s+t)^2}\right)|b-c|^2}. \end{aligned}$$

One of these cases must occur, for if $|b-c| \geq s+t$, then

$$|b-c| > \max\{s,t\} > |s-t|,$$

since $s, t > 0$. However, we could easily have both cases simultaneously. For instance, the pair S^n and $S(e_1, \frac{1}{2})$ satisfies both conditions, and these are inverses in both $S\left(2e_1, \sqrt{\frac{3}{2}}\right)$ and $S\left(\frac{2}{3}e_1, \sqrt{\frac{5}{18}}\right)$, as can now easily be checked.

The following theorem summarizes these results.

Theorem 20 *There is at least one Möbius transformation mapping any sphere to another. In particular, $S(b,s)$ is mapped to $S(c,t)$ via inversion in:*

- *the plane*
$$P\left(\frac{1}{2}(a-b), \frac{1}{4}|a|^2 - \frac{1}{4}|b|^2\right),$$
if $s = t$;

- *the sphere centered at*
$$a = \left(\frac{s}{s-t}\right)c - \left(\frac{t}{s-t}\right)b$$
with radius
$$r = \sqrt{\left(\frac{ts}{(s-t)^2}\right)|b-c|^2 - ts},$$
if $s \neq t$ and $|b-a| > |s-t|$;

- *the sphere centered at*
$$a = \left(\frac{s}{s+t}\right)c + \left(\frac{t}{s+t}\right)b$$
with radius
$$r = \sqrt{ts - \left(\frac{ts}{(s+t)^2}\right)|b-c|^2},$$
if $|b-c| < s+t$.

Planes reflected in spheres

Occasionally it is useful to know the general form a plane inverted in a sphere. We derive it here, with a method similar to that used to find the inverse of a sphere.

Let $P(b,t)$ be any plane and $S(a,r)$ be any sphere. To invert the plane in the sphere, we subject it to the four transformations outlined above:

1. Translation by $-a$ produces $P(b, t - (a \cdot b))$.

2. Since the plane now has equation
$$-2(x \cdot b) + 2t - 2(a \cdot b) = 0,$$
inverting in the unit sphere gives
$$-2(x \cdot b) + [2t - 2(a \cdot b)]\,|x|^2 = 0, \tag{5}$$
by (2).

(a) When $a \notin P(b,t)$ (i.e. $2t - 2(a \cdot b) \neq 0$), (5) becomes

$$|x|^2 - \left(\frac{1}{t - (a \cdot b)}\right)(x \cdot b) = 0,$$

which is the equation for

$$S\left(\left(\frac{1}{2t - 2(a \cdot b)}\right) b, \frac{|b|}{2\left|t - (a \cdot b)\right|}\right).$$

(b) When $a \in P(b,t)$, (5) becomes

$$(x \cdot b) = 0,$$

the equation for $P(b,0)$.

3. (a) Expansion by r^2 gives

$$S\left(\left(\frac{r^2}{2t - 2(a \cdot b)}\right) b, \frac{r^2|b|}{2\left|t - (a \cdot b)\right|}\right).$$

(b) The plane $P(b,0)$ is invariant under expansion.

4. (a) Translating by a gives

$$S\left(a + \left(\frac{r^2}{2t - 2(a \cdot b)}\right) b, \frac{r^2|b|}{2\left|t - (a \cdot b)\right|}\right)$$

as a final result when $a \notin P(b,t)$.

(b) Otherwise, we have

$$P\left(b, (a \cdot b)\right).$$

Reflections in planes

To be complete, we should also find the images of generic planes and spheres when reflected across a given plane. One of these is easy to find; since reflection across a plane preserves Euclidean distance, the image of $S(a,r)$ across $P(b,t)$ is

$$\begin{aligned}\phi\left(S(a,r)\right) &= S\left(\phi(a), r\right)\\ &= S\left(a - \left(\frac{2\left[(a \cdot b) - t\right]}{|b|^2}\right) b, r\right).\end{aligned}$$

Finding the image of a plane will require more work. Using theorem 7, write reflection across $P(b,s)$ as

$$\phi(x) = Ax + x_0,$$

where A is an orthogonal matrix and $x_0 = \phi(0)$. Observe that

$$\begin{aligned}
|\phi(x) - \phi(a)|^2 &= |(Ax + x_0) - \phi(a)|^2 \\
&= |Ax + x_0|^2 - 2(Ax + x_0 \cdot \phi(a)) + |\phi(a)|^2 \\
&= |Ax|^2 + 2(Ax \cdot x_0) + |x_0|^2 - 2(Ax \cdot \phi(a)) - 2(x_0 \cdot \phi(a)) + |\phi(a)|^2 \\
&= |x|^2 + 2(Ax \cdot x_0) + |x_0|^2 - 2(Ax \cdot \phi(a)) - 2(x_0 \cdot \phi(a)) + |\phi(a)|^2 \\
&= |x|^2 + 2(Ax \cdot x_0 - \phi(a)) + |x_0|^2 - 2(x_0 \cdot \phi(a)) + |\phi(a)|^2.
\end{aligned}$$

Since

$$\begin{aligned}
2(Ax \cdot x_0 - \phi(a)) &= 2(Ax + x_0 \cdot x_0 - \phi(a)) - 2(x_0 \cdot x_0 - \phi(a)) \\
&= 2\left(\phi(x) \cdot x_0 - \phi(a)\right) - 2|x_0|^2 + 2(x_0 \cdot \phi(a)),
\end{aligned}$$

we have

$$|\phi(x) - \phi(a)|^2 = |x|^2 + 2\left(\phi(x) \cdot x_0 - \phi(a)\right) - |x_0|^2 + |\phi(a)|^2.$$

Reflection in a sphere preserves distance, so $|\phi(x) - \phi(a)|^2 = |x - a|^2$, and

$$\begin{aligned}
|x|^2 + 2\left(\phi(x) \cdot x_0 - \phi(a)\right) - |x_0|^2 + |\phi(a)|^2 &= |x|^2 - 2(x \cdot a) + |a|^2 \\
2\left(\phi(x) \cdot x_0 - \phi(a)\right) &= 2(x \cdot a) + |x_0|^2 + |a|^2 - |\phi(a)|^2 \\
\left(\phi(x) \cdot x_0 - \phi(a)\right) &= (x \cdot a) + \frac{1}{2}|x_0|^2 + \frac{1}{2}|a|^2 - \frac{1}{2}|\phi(a)|^2.
\end{aligned}$$

Now

$$\begin{aligned}
|\phi(a)|^2 &= \left| a - \left(\frac{2\left[(a \cdot b) - s\right]}{|b|^2} \right) b \right|^2 \\
&= |a|^2 - \left(\frac{4\left[(a \cdot b) - s\right]}{|b|^2} \right)(a \cdot b) + \frac{4\left[(a \cdot b) - s\right]^2}{|b|^2},
\end{aligned}$$

and

$$\begin{aligned}
|x_0|^2 &= |\phi(0)|^2 \\
&= |\frac{2s}{|b|^2} b|^2 \\
&= \frac{4s}{|b|^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}(\phi(x)\cdot\phi(0)-\phi(a)) &= (x\cdot a)+\frac{2s}{|b|^2}+\frac{2}{|b|^2}\left[(a\cdot b)^2-s(a\cdot b)\right]-\frac{2}{|b|^2}\left[(a\cdot b)^2-2s(a\cdot b)+s^2\right]\\ &= (x\cdot a)+2s(a\cdot b).\end{aligned}$$

Since

$$\begin{aligned}\phi(0)-\phi(a) &= \frac{2s}{|b|^2}b-a+\left(\frac{2\left[(a\cdot b)-s\right]}{|b|^2}\right)b\\ &= \left(\frac{2(a\cdot b)}{|b|^2}\right)b-a,\end{aligned}$$

we've found the answer we're looking for.

Theorem 21 *$P(a,t)$ is mapped by inversion in $P(b,s)$ to*

$$P\left(\left(\frac{2(a\cdot b)}{|b|^2}\right)b-a,t+4s(a\cdot b)\right).$$

Conclusion

We've seen that any sphere can be mapped onto any other by a Möbius transformation, and we've also seen that any plane can be mapped onto a sphere by an inversion. When given a plane and a sphere, we can therefore find a Möbius transformation that maps each onto the other. Taken with the other results in this appendix, this proves a much more general result.

Theorem 22 *If (Σ,Σ') is any combination of planes and spheres, then there is some Möbius transformation ϕ such that $\phi(\Sigma)=\Sigma'$.*

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