A VERY BRIEF OVERVIEW OF LIE ALGEBRAS

STEPHANIE KERNIK

COURSE: MATH 4901
FACULTY ADVISER: MICHAEL O'REILLY
UNIVERSITY OF MINNESOTA, MORRIS
SPRING 2008

ABSTRACT. The study of Lie algebras can be a powerful tool in mathematics and physics. This paper makes an attempt to demonstrate in part why this is so. In this paper, basic concepts concerning Lie algebras are defined, explained, and explored. Some fundamental theorems, including Engel's Theorem, Lie's Theorem, and Cartan's Criteria are stated and proved. It is shown that a Lie algebra can be written as a direct sum of the maximal solvable ideal and a semisimple algebra that can in turn be decomposed further into simple ideals; some useful consequences of this fact are discussed. Finally, this paper briefly touches on representation theory in Lie algebras by considering some aspects of roots and weights. By the end of the paper, a method is outlined to determine the structure of a Lie algebra, which characterizes completely the local structure of the Lie group with which it is associated.

1. INTRODUCTION

The study of Lie theory began in the last third of the nineteenth century, motivated by the mathematician Sophus Lie's work on finding solutions to partial differential equations. He employed a novel approach, combining transformations that preserve a type of geometric structure (specifically, a contact structure) and group theory to arrive at a theory of continuous transformation groups [1]. Since then, Lie Theory has been found to have applications in many areas of mathematics, including the study of special functions, differential and algebraic geometry, number theory, group and ring theory, and topology [2, 3, 4]. It has also become instrumental in parts of physics, for some Lie algebras arise naturally from symmetries in physical systems, and is a powerful tool in such areas as classical and quantum mechanics, atomic spectroscopy, solid state physics, and elementary particles [2, 3, 5]. Thus, there is ample reason to study Lie theory.

It is the intent of this paper to give a very brief introduction to the study of Lie algebras. We will prove some fundamental theorems and eventually show that a Lie algebra can be written as a direct sum of solvable and simple Lie algebras (terms that will be defined below). As Lie algebras have been found to be so useful to physics, an attempt will be made to provide physically relevant examples at all stages of the paper. We will end with a short discussion on further techniques in studying the structure of a Lie algebra that could not be included in the paper in full. First, however, we will consider some aspects of Lie groups, to better motivate the study of their corresponding Lie algebras.

2. LIE GROUPS

In a very general sense, a Lie group can be thought of as a group of continuous symmetries. It is this connection with symmetry that makes them so practical and prevalent in physics. For example, the symmetry scheme used to describe elementary particles called *hadrons* and their interactions uses a Lie group [5]. Lie groups are also extremely useful tools in many areas of mathematics [3]. In this section, we will define *Lie group*, list a few examples, and see why one would want to study Lie algebras instead. Note that this section is intended only to provide motivation for the rest of the paper; there are an abundance of books on Lie groups for readers who are interested in more details.

Definition 2.1. *Lie Group:* A **Lie group** is a set of points which satisfies three properties:

1. It is *locally Euclidean* (that is, for every x in the set, there exists a neighborhood of x that is homeomorphic to an open set of $\mathbb{R}^n$)
2. It has a group structure
3. Its group operations are differentiable

Lie groups can be considered as an abstract group, a topological space, or an analytic manifold. J. G. Belinfante et al eloquently describe Lie groups as "the structure which naturally results when analytic flesh is put on the bones of abstract group theory" [2].

Some simple examples of Lie groups include the real line under addition and the circle once a specific point is designated as the identity [3]. Matrix groups with multiplication as the group operation are also Lie groups. Consider, for instance, the general linear group $GL(V)$, the $n \times n$ invertible matrices. Note that the matrices in the matrix group must be nonsingular and of a fixed size so that the group operation is defined and inverses exist for each element [2].

A problem with Lie groups is that they can be difficult to work with and understand [6]. However, the structure of a Lie group gives us a powerful advantage: most of the information of the global structure of the group is determined by its local structure, i.e, what happens in an arbitrarily small coordinate neighborhood. Without loss of generality, one needs only to study what happens near the identity to get information about the whole group [2].

Here is where Lie algebras become relevant. Each Lie group has a corresponding Lie algebra which completely determines the local structure of the group [2]. Thus, characterizing the structure of the Lie algebra will tell us helpful information about the Lie group. Due to the fact that a Lie algebra is merely a bilinear operation on a vector space, it is usually much easier to define and use than the concept of a Lie group [3]. We can see, then, that one reason to study Lie algebras is to better understand ideas described by Lie groups. With this motivation in mind, we will now focus on Lie algebras.

3. LIE ALGEBRAS

In this section, we will develop the basic concepts we need to understand aspects of Lie algebras and move toward showing that a Lie algebra can be decomposed into Lie algebras that are simpler to work with. We will define the term 'Lie Algebra' and look at several examples. We will also consider several basic definitions from abstract algebra in terms of Lie algebras and we will introduce the idea of *solvability* and *nilpotence*. Finally, we will prove a fundamental theorem.

Definition. *Algebra:* An **algebra** is a vector space V over a field $\mathbb{F}$ with an operation $\circ$ which satisfies the following: For every $\alpha \in \mathbb{F}$ and $x, y, z \in V$,

i) $(x+y) \circ z = x \circ z + y \circ z$
ii) $x \circ (y+z) = x \circ y + x \circ z$
iii) $\alpha(x \circ y) = (\alpha x) \circ y = x \circ (\alpha y)$

Definition 3.1. *Lie Algebra:* A **Lie algebra** consists of a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ with an operation $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : \ (x, y) \mapsto [x, y]$, called the *bracket* of x and y, that has the following properties:

i) *(Bilinearity)* $[ax + by, z] = a[x, z] + b[y, z]$ for all $a, b \in \mathbb{F}$ and all $x, y, z \in \mathfrak{g}$.
ii) *(Skew Symmetry)* $[x, x] = 0$ for all $x \in \mathfrak{g}$.
iii) *(Jacobi Identity)* $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Note that (i) and (ii) combined imply anticommitativity:

$$0 \ = \ [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x] \Rightarrow [x, y] = -[y, x].$$

Notice further that the vector space underlying a Lie algebra can be infinite dimensional; however, in this paper we will consider *only* Lie algebras that are finite dimensional.

Now let us consider some simple examples.

Example 3.2. *Abelian Lie Algebra:* Let L be a arbitrary finite dimensional vector space over $\mathbb{F}$ and set $[x, y] = 0$ for all $x, y \in L$. It is obvious that L is a Lie algebra:

i) Bilinearity: For all $a, b \in \mathbb{F}$ and all $x, y, z \in A$, $[ax + by, z] = 0$ and $a[x, z] + b[y, z] = 0$.
ii) Skew Symmetry: Clearly, $[x, x] = 0$.
iii) Jacobi Identity: For all $x, y, z \in L$, $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 + 0 + 0 = 0$.

This sort of Lie algebra is called **abelian**. The concept of an abelian algebra will return later in this paper.

Example 3.3. *The Trivial Lie Algebra Construction:* Let A be a vector space over a field $\mathbb{F}$ with an associative multiplication $x \cdot y$ for $x, y \in A$. Define the bracket as $[x, y] = x \cdot y - y \cdot x$ and A becomes a Lie algebra. It is easy to check:

i) Bilinearity: For all $a, b \in \mathbb{F}$ and all $x, y, z \in A$, $[ax + by, z] = (ax + by)z - z(ax + by) = axz - azx + byz - bzy = a[x, z] + b[y, z]$.
ii) Skew Symmetry: Clearly, $[x, x] = 0$.
iii) Jacobi Identity:

$$\begin{aligned} [x, [y, z]] + [y, [z, x]] + [z, [x, y]] &= xyz - xzy - yzx + zyx + yzx - yxz - zxy \\ &\qquad +xzy + zxy - zyx - xyz + yxz \\ &= xyz - xyz - xzy + xzy - yzx + yzx + zyx \\ &\qquad -zyx - yxz + yxz - zxy + zxy \\ &= 0 \end{aligned}$$

Using this example, we can now define a Lie algebra that is quite important.

Definition 3.4. *General Linear Algebra:* Consider the set of linear transformations $V \to V$, where V is a finite dimensional vector space over some field $\mathbb{F}$. Define the bracket on this set as in the above example, $[x, y] = xy - yx$. This algebra is called the **general linear (Lie) algebra**, denoted $\mathfrak{gl}(V)$.

An immediately useful idea comes from abstract algebra: subalgebras. In the interest of completeness, we will define the term in the context of Lie algebras, using the bracket. Following the standard definition, a subspace $\mathfrak{h}$ is a (Lie) **subalgebra** of a Lie algebra $\mathfrak{g}$ if $[x, y] \in \mathfrak{h} \ \ \forall x, y \in \mathfrak{h}$. As the three properties of a Lie algebra are all inherited, $\mathfrak{h}$ is a Lie algebra in its own right.

A *linear Lie algebra* is a subalgebra of the general linear algebra, defined above. Linear Lie algebras are fundamental to the study of Lie algebras, but not relevant to the small section covered in this paper. Thus, we will not discuss them further. Due to the fact that nearly all Lie algebras used in physics are linear, most of the examples used in this paper are linear Lie algebras [7].

3.1. **Algebras in this paper.**

In this subsection, we will define two algebras that we are going to use extensively as examples of concepts. Both have many applications in physics.

The special unitary Lie algebra, $\mathfrak{su}(2)$:

This is a linear Lie algebra. Use the basis a_1, a_2, a_3 with

$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad a_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad a_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The commutation relations are easily calculated as

$$[a_1, a_2] = -a_3, \quad [a_2, a_3] = -a_1, \quad [a_3, a_1] = -a_2,$$

which can also be written as $[a_i, a_j] = \epsilon_{ijk} a_k$ where

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1),\ (3,1,2) \\ -1 & \text{if } (i,j,k) = (2,1,3),\ (1,3,2),\ (3,2,1) \\ 0 & \text{otherwise} \end{cases}.$$

An example of this in physics comes from quantum mechanics in three dimensions. The commutation relations of the three angular momentum operators (one for each dimension) generate $\mathfrak{su}(2)$ [5].

The Heisenberg algebra, $\mathfrak{hei}(n)$:

Consider a basis $r_1, \dots r_n, p_1, \dots p_n, z$. The commutation relations are defined as

$$[r_i, r_j] = [p_i, p_j] = [r_i, z] = [p_i, z] = [z, z] = 0 \text{ and } [r_i, p_j] = \delta_{ij} z \text{ for } 1 \le i, j \le n.$$

Note that δ_{ij} is the Kronecker delta, defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

If we let $n = 3$, take the basis vectors to be operators on a Hilbert space, and adjust by a constant so that $[r_i, p_j] = -i\hbar\delta_{ij} z$ (where $\hbar$ is Planck's constant), we get Heisenberg's famous uncertainty principle [5].

3.2. **Basic Algebraic Concepts.**

We have already seen the definition of a (Lie) subalgebra above; we will now consider other standard algebraic ideas in terms of Lie algebras.

3.2.1. *Ideals.*

A subspace $\mathfrak{q}$ of a Lie algebra $\mathfrak{g}$ is an **ideal** of $\mathfrak{g}$ if it has the property that whenever $x \in \mathfrak{g}$ and $y \in \mathfrak{q}$, $[x, y] \in \mathfrak{q}$. Ideals in a Lie algebra act like normal subgroups in group theory: using them, one can analyze the structure of a Lie algebra and one can construct quotient algebras [4].

Let us consider some examples. Clearly, the subspace containing the zero vector (henceforth denoted $\mathbf{0}$) and $\mathfrak{g}$ itself are both ideals of $\mathfrak{g}$. The **center**, $Z(\mathfrak{g}) = \{z \in \mathfrak{g} | [x, z] = 0 \ \ \forall x \in \mathfrak{g}\}$, is also an ideal of $\mathfrak{g}$. Recall that a Lie algebra is abelian if $[x, y] = 0$; it is evident that a Lie algebra $\mathfrak{g}$ is abelian iff $Z(\mathfrak{g}) = \mathfrak{g}$. Another example of a nontrivial ideal is the **derived algebra** of $\mathfrak{g}$, which consists of all linear combinations of $[x, y]$ ($x, y \in \mathfrak{g}$) and is denoted $[\mathfrak{g}, \mathfrak{g}]$. We can see it is an ideal because $[a, b]$ will be some linear combination of $[x, y]$ ($x, y \in \mathfrak{g}$) for every $a \in \mathfrak{g}$ and $b \in [\mathfrak{g}, \mathfrak{g}]$. The derived algebra can also determine if a Lie algebra is abelian or not: $[\mathfrak{g}, \mathfrak{g}] = \mathbf{0}$ implies that $[x, y] = 0 \ \forall x, y \in \mathfrak{g}$; thus a Lie algebra $\mathfrak{g}$ is abelian iff its derived algebra is the zero vector.

Just as there exists a concept of a simple group in group theory, there is a concept of a simple algebra in the theory of Lie algebras. A Lie algebra $\mathfrak{g}$ is **simple** if it has no ideals except itself and $\mathbf{0}$, and $[\mathfrak{g},\mathfrak{g}] \neq \mathbf{0}$, that is, $\mathfrak{g}$ is not abelian. If $\mathfrak{g}$ is simple, then $Z(\mathfrak{g}) = \mathbf{0}$ and $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. This can be easily seen: If $\mathfrak{g}$ is simple, then $\mathbf{0}$ and $\mathfrak{g}$ are its only ideals. Since $Z(\mathfrak{g})$ and $[\mathfrak{g},\mathfrak{g}]$ are also ideals, both must either be $\mathbf{0}$ or $\mathfrak{g}$. However, $\mathfrak{g}$ is not abelian, so $Z(\mathfrak{g}) \neq \mathfrak{g}$ and $[\mathfrak{g},\mathfrak{g}] \neq \mathbf{0}$.

Example 3.5. $\mathfrak{su}(2)$ *is simple:* Recall the commutation relations found for $\mathfrak{su}(2)$ in Subsec. 3.1: $[a_i, a_j] = \epsilon_{ijk}a_k$ for the basis a_1, a_2, a_3. Suppose $\mathfrak{q}$ is an ideal of $\mathfrak{g}$; then $[x, y] \in \mathfrak{q}$ for every $x \in \mathfrak{g}$ and $y \in \mathfrak{q}$. The subspace $\mathfrak{q}$ will be spanned by some combination of (a_1, a_2, a_3); there are 8 possibilities. ($\mathbf{0}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \ldots, \{a_1, a_2, a_3\}$) Let us suppose $a_1 \in \mathfrak{q}$. Since $[a_2, a_1] = a_3$, a_3 must be in $\mathfrak{q}$, and since $[a_1, a_3] = a_2$, this implies that $a_2 \in \mathfrak{q}$ as well. Similar calculations show that if one element is in $\mathfrak{q}$, then the rest must also be there. Thus, the only possible ideals are spanned by $\mathbf{0}$ or (a_1, a_2, a_3), the second of which is $\mathfrak{g}$; $\mathfrak{su}(2)$ is simple.

When a Lie algebra $\mathfrak{g}$ is *not* simple, one can 'factor out' a nonzero proper ideal $\mathfrak{q}$ to get a Lie algebra of smaller dimension; this is a **quotient algebra**, denoted $\mathfrak{g}/\mathfrak{q}$. We can construct a quotient algebra like one constructs a quotient ring. Define a equivalence relation $\sim$ on $\mathfrak{g}$ so that $a \sim b$ iff $b - a \in \mathfrak{q}$. Notate the equivalence class of a as $a + \mathfrak{q}$ and denote the set of all equivalence classes as $\mathfrak{g}/\mathfrak{q}$. Then $\mathfrak{g}/\mathfrak{q}$ is the quotient space, and we can define Lie multiplication as $[a + \mathfrak{q}, b + \mathfrak{q}] = [a, b] + \mathfrak{q}$ to make $\mathfrak{g}/\mathfrak{q}$ into a quotient algebra. Note that there is no uncertainly in the definition of Lie multiplication: if $a + \mathfrak{q} = a' + \mathfrak{q}$ and $b + \mathfrak{q} = b' + \mathfrak{q}$, then we can write $a' = a + u$ ($u \in \mathfrak{q}$), $b' = b + v$ ($v \in \mathfrak{q}$) and see that

$$[a', b'] = [a + u, b + v] = [a + u, b] + [a + u, v] = [a, b] + [u, b] + [a, v] + [u, v].$$

Since $[u, b]$, $[a, v]$, and $[u, v]$ are all elements of $\mathfrak{q}$, we can see that $[a', b'] + \mathfrak{q} = [a, b] + \mathfrak{q}$.

A final fundamental concept of ideals and Lie algebras is the *direct sum*. A Lie algebra $\mathfrak{g}$ is a **direct sum** of its ideals $\mathfrak{q}, \ldots, \mathfrak{q}_t$ when $\mathfrak{g}$ is made up of the vector space direct sum of the subspaces $\mathfrak{q}, \ldots, \mathfrak{q}_t$: $\mathfrak{g} = \mathfrak{q} + \ldots + \mathfrak{q}_t$. This can be viewed as defining the Lie products 'componentwise', with $[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2])$. Note that $[\mathfrak{q}_i, \mathfrak{q}_j] \subset \mathfrak{q}_i \cap \mathfrak{q}_j = 0$ if $i \neq j$. We write $\mathfrak{g} = \mathfrak{q} \oplus \ldots \oplus \mathfrak{q}_t$.

3.2.2. *Homomorphisms.* Just as one would expect, a **homomorphism** is a linear transformation $\phi : \mathfrak{g} \to \mathfrak{g}'$ ($\mathfrak{g}$,$\mathfrak{g}'$ Lie algebras) where $\phi([x, y]) = [\phi(x), \phi(y)]\ \forall x, y \in \mathfrak{g}$. Furthermore, ϕ is a **monomorphism** if $\ker\phi = \mathbf{0}$, an **epimorphism** if $\operatorname{im}\phi = \mathfrak{g}'$, and an **isomorphism** if ϕ is both. An **automorphism** is an isomorphism of a Lie algebra with itself.

Example 3.6. *The algebras* $\mathfrak{su}(2)$ *and* $\mathfrak{so}(3)$ *are isomorphic:*

Once again, recall $\mathfrak{su}(2)$ from Subsec. 3.1, in which we found the commutation relations to be $[a_i, a_j] = -\epsilon_{ijk}a_k$ for the basis a_1, a_2, a_3. Let us turn to $\mathfrak{so}(3)$, another linear Lie algebra. We can fix a basis x_1, x_2, x_3 for $\mathfrak{so}(3)$, with

$$x_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},\ x_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ x_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Computing, one can find that $[x_1, x_2] = -x_3$, $[x_2, x_3] = -x_1$, and $[x_3, x_1] = -x_2$ which can be written as $[x_i, x_j] = -\epsilon_{ijk}x_k$. Next, let $\mathfrak{g} = \mathfrak{su}(2)$ and $\mathfrak{g}' = \mathfrak{so}(3)$ and define $\phi : \mathfrak{g} \to \mathfrak{g}'$ such that $a_i \mapsto x_i$ for all $a_i \in \mathfrak{g}$ and $x_i \in \mathfrak{g}'$. It can be shown that ϕ is a homomorphism: $\phi([a_i, a_j]) = \phi(-\epsilon_{ijk}a_k) = -\epsilon_{ijk}x_k = [x_i, x_j] = [\phi(a_i), \phi(a_j)]$. It is also easy to see that $\ker\phi = 0$, and $\operatorname{im}\phi = \mathfrak{g}'$. Thus, ϕ is an isomorphism and $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ are isomorphic.

Note that $\ker\phi$ is an ideal of $\mathfrak{g}$: if $\phi(x)=0$, then $\phi([x,y])=[\phi(x),\phi(y)]=0\ \ \forall y\in\mathfrak{g}$. Note also that $\operatorname{im}\phi$ is a subalgebra of $\mathfrak{g}'$: consider $x,y\in\operatorname{im}\phi$. This means $\phi(a)=x$ and $\phi(b)=y$ for some $a,b\in\mathfrak{g}$. Since $[x,y]=[\phi(a),\phi(b)]=\phi([a,b])$ and $[a,b]\in\mathfrak{g}$, we can see that $[x,y]\in\operatorname{im}\phi$.

3.2.3. *Representations.* A **representation** of a Lie algebra $\mathfrak{g}$ is a homomorphism $\rho:\mathfrak{g}\to\mathfrak{gl}(V)$, where V is some vector space over $\mathbb{F}$. For $V=\mathbb{F}^n$, the operators $\rho(x)$ $(x\in\mathfrak{g})$ are matrices. The *trivial representation* is the representation ρ which is defined by $\rho(x)=(\mathbf{0})$ $(x\in\mathfrak{g})$.

An extremely important example for the study of Lie algebras is the *adjoint representation* $\mathrm{ad}:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$, where $\mathrm{ad}\,x(y)=[x,y]\ \ \forall x,y\in\mathfrak{g}$. The adjoint representation is clearly a linear transformation; it also preserves the bracket:

$$\begin{aligned}
[\mathrm{ad}\,x,\mathrm{ad}\,y](z) &= \mathrm{ad}\,x\,\mathrm{ad}\,y(z)-\mathrm{ad}\,y\,\mathrm{ad}\,x(z)\\
&= \mathrm{ad}\,x\,[y,z]-\mathrm{ad}\,y\,[x,z]\\
&= [x,[y,z]]+[[x,z],y]\\
&= [x,[y,z]]-[[z,x],y] && \text{(Property (ii) of a Lie algebra)}\\
&= [[z,x],y]+[[x,y],z]-[[z,x],y] && (\,[x,[y,z]]=[[z,x],y]+[[x,y],z]\text{ by (iii)})\\
&= [[x,y],z]\\
&= \mathrm{ad}\,[x,y](z)
\end{aligned}$$

The kernel of ad is comprised of all $x\in\mathfrak{g}$ for which $[x,y]=0$ (for all $y\in\mathfrak{g}$); thus $\ker\mathrm{ad}\,x=Z(\mathfrak{g})$.

Example 3.7. *The adjoint representations of* $\mathfrak{su}(2)$*:* Recall from Subsec. 3.1 the commutation relations that characterize $\mathfrak{su}(2)$: $[a_1,a_2]=-a_3$, $[a_2,a_3]=-a_1$, and $[a_3,a_1]=-a_2$ over the basis a_1,a_2,a_3. The adjoint representation of each element is

$$\mathrm{ad}\,a_1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\quad \mathrm{ad}\,a_2=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\quad \mathrm{ad}\,a_3=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

Representations play an important role in classifying ideals. The adjoint representation has a particular usefulness. The short discussion in Sec. 5 will touch on this idea, but the powerful realm of representations of Lie algebras is sadly beyond the scope of this paper.

3.3. Nilpotence and Solvability.

We will use the concepts of *solvable* and *nilpotent* Lie algebras to further study and classify the ideals that make up a Lie algebra. Due to the fact that ideals play the same role for Lie algebras as normal subgroups play in group theory, an understanding of the ideals is an important step toward reducing the Lie algebra to something easier to study [2]. In this subsection, we will encounter the definitions of the terms ‘solvable’ and ‘nilpotent’, and prove our first major theorem (Engel’s Theorem). We will also define the term ‘semisimple’, whose importance will become apparent in the next section.

3.3.1. *Solvable Lie Algebras.* Recall the definition of a *derived algebra* from sub-subsec. 3.2.1. Define a sequence of ideals of a Lie algebra $\mathfrak{g}$ by

$$\begin{aligned} \mathfrak{g}^{(0)} &= \mathfrak{g} \\ \mathfrak{g}^{(1)} &= [\mathfrak{g}, \mathfrak{g}] \\ \mathfrak{g}^{(2)} &= [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \\ &\vdots \\ \mathfrak{g}^{(i)} &= [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}] \end{aligned}$$

This is called the **derived series**. The derived series can be thought of as the fastest descending series whose quotients are abelian [8]. We refer to $\mathfrak{g}$ as **solvable** if the derived series goes down to zero, i.e., $\mathfrak{g}^{(n)} = \mathbf{0}$ for some $n \in \mathbb{N}$. Any abelian Lie algebra is solvable: recall that $\mathfrak{g}$ is abelian iff $[\mathfrak{g}, \mathfrak{g}] = \mathbf{0}$. It is also easy to see that any simple algebra is nonsolvable because in the simple situation, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and the derived series becomes

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g},\ \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \mathfrak{g},\ \ldots$$

Clearly, there is no n such that $\mathfrak{g}^{(n)} = \mathbf{0}$.

Example 3.8. *The Heisenberg Algebra is solvable:* Recall the Heisenberg algebra from Subsec. 3.1. Suppose $\mathfrak{g}$ is a Heisenberg algebra with the basis $r_1, \ldots r_n, p_1, \ldots p_n, z$; $\mathfrak{g}$ has the commutation relations $[r_i, r_j] = [p_i, p_j] = [r_i, z] = [p_i, z] = [z, z] = 0$ and $[r_i, p_j] = \delta_{ij} z$ for $1 \leq i, j \leq n$. Now we construct the derived series. It is easy to see that $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \{z\}$. Continuing, we can find that $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \mathbf{0}$, and therefore the Heisenberg algebra is solvable.

Now that we have a definition of solvable, we can state three simple yet useful facts about solvability. The proofs are provided in Appendix A on page 18.

Proposition 3.9.

- i) Suppose $\mathfrak{g}$ is a solvable Lie Algebra. Then all subalgebras and homomorphic images of $\mathfrak{g}$ are also solvable.
- ii) Suppose $\mathfrak{q}$ is a solvable ideal of a Lie algebra $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{q}$ is solvable. Then $\mathfrak{g}$ is solvable as well.
- iii) Suppose $\mathfrak{q}$ and $\mathfrak{r}$ are solvable ideals of a Lie algebra $\mathfrak{g}$. Then $\mathfrak{q} + \mathfrak{r}$ is likewise solvable.

Next we come a definition of fundamental importance in the study of Lie algebras, the *semisimple* Lie algebra [9]. To define this, we will first need to show that every Lie algebra has a unique solvable maximal ideal. Let $\mathfrak{g}$ be an arbitrary Lie algebra, $\mathfrak{r}$ be an ideal included in no larger solvable ideal, and $\mathfrak{q}$ be any other solvable ideal of $\mathfrak{g}$. Maximality and Prop. 3.9 mean that $\mathfrak{r}+\mathfrak{q} = \mathfrak{r}$, that is, $\mathfrak{q} \subset \mathfrak{r}$; thus $\mathfrak{r}$ is unique. The unique maximal solvable ideal is called the **radical** of $\mathfrak{g}$ and is denoted $\operatorname{rad} \mathfrak{g}$. Armed with this, we can now proceed to a definition: *a Lie algebra is **semisimple** if* $\operatorname{rad} \mathfrak{g} = \mathbf{0}$. Note that $Z(\mathfrak{g}) = \mathbf{0}$ for a semisimple Lie algebra; $Z(\mathfrak{g})$ is an abelian ideal, and since abelian algebras are solvable, $Z(\mathfrak{g})$ must be zero by definition of semisimple.

We can show that a simple algebra $\mathfrak{g}$ is also semisimple: its only ideals are $\mathbf{0}$ and $\mathfrak{g}$ and, being unsolvable, $\mathfrak{g} \neq \mathbf{0}$. Thus, $\operatorname{rad} \mathfrak{g} = \mathbf{0}$. The converse is not true.

Example 3.10. $\mathfrak{g} = \mathfrak{so}(4)$ *is semisimple but not simple:* Define a basis $a_1, a_2, a_3, b_1, b_2, b_3$ for the linear Lie algebra $\mathfrak{so}(4)$ as

$$a_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad a_3 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$b_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad b_2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad b_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

One can find that $[a_i, a_j] = -\epsilon_{ijk}a_k$, $[b_i, b_j] = -\epsilon_{ijk}b_k$ and $[a_i, b_j] = 0$ for $i, j = 1, 2, 3$. Let $\mathfrak{q}_a$ be the subalgebra of $\mathfrak{g}$ spanned by (a_1, a_2, a_3). Returning to the commutation relations above, we can see that $\mathfrak{q}_a$ is an ideal of $\mathfrak{g}$. Similarly, $\mathfrak{q}_b$, spanned by (b_1, b_2, b_3), is also an ideal of $\mathfrak{g}$. Thinking a little more, we can see that the only other ideals of $\mathfrak{g}$ are $\mathbf{0}$ and itself. The presence of the ideals $\mathfrak{q}_a$ and $\mathfrak{q}_b$ mean that $\mathfrak{g}$ is not simple. However, $\mathfrak{g}$ is semisimple: it is easily found that $\mathfrak{q}_a^{(1)} = [\mathfrak{q}_\mathfrak{a}, \mathfrak{q}_\mathfrak{a}] = \mathfrak{q}_a$ and $\mathfrak{q}_b^{(1)} = [\mathfrak{q}_\mathfrak{b}, \mathfrak{q}_\mathfrak{b}] = \mathfrak{q}_b$; this means neither is solvable. Thus we can see that a semisimple Lie algebra is not necessarily a simple one.

Another notable property is that $\mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ is semisimple for any arbitrary Lie algebra $\mathfrak{g}$. This follows from Prop. 3.9. Suppose $\mathfrak{q} \in \mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ is a solvable ideal. Then $\pi^{-1}(\mathfrak{q})$ is a solvable ideal (Prop. 3.9), where π is the usual canonical map. We can find the projection $\mathrm{rad}\,\mathfrak{g} \to \pi^{-1}(\mathfrak{q}) \to \mathfrak{q} \to \mathfrak{g}$, and now we have a contradiction! By definition, $\mathrm{rad}\,\mathfrak{g}$ is the largest solvable ideal; $\pi^{-1}(\mathfrak{q})$ cannot be larger. Therefore $\mathfrak{q}$ does not exist and $\mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ is semisimple. This implies every Lie algebra is built out of a semisimple algebra and a solvable algebra [9]. In fact, Eugenio Levi showed that every Lie algebra is a direct sum of a radical and a semisimple ideal [2]. We have succeeded in decomposing the Lie algebra, and we will see in Sec. 4 that it can be decomposed further. Before this, though, we must gather a few more definitions and theorems.

3.3.2. *Nilpotent Lie Algebras.* Once again, we will begin by defining a sequence of ideals of $\mathfrak{g}$. This sequence is often called the **descending central series**.

$$\begin{aligned} \mathfrak{g}^0 &= \mathfrak{g} \\ \mathfrak{g}^1 &= [\mathfrak{g}, \mathfrak{g}] \\ \mathfrak{g}^2 &= [\mathfrak{g}, \mathfrak{g}^1] \\ &\vdots \\ \mathfrak{g}^i &= [\mathfrak{g}, \mathfrak{g}^{i-1}] \end{aligned}$$

We call $\mathfrak{g}$ **nilpotent** if $\mathfrak{g}^n = \mathbf{0}$ for some $n \in \mathbb{N}$. Any abelian Lie algebra is nilpotent: $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \mathbf{0}$. More interestingly, $\mathfrak{g}^{(i)} \subset \mathfrak{g}^i\ \forall i$, so all nilpotent Lie algebras are solvable [4]. The converse is not true; for an instance of this, see Ex. 3.12 on the next page. There is another useful connection concerning the derived algebra: if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, then $\mathfrak{g}$ is solvable. This is quite easy to see. If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, then it is solvable. Suppose $[\mathfrak{g}, \mathfrak{g}]^{(k)} = \mathbf{0}$ for some k. Then, $(\mathfrak{g}^{(1)})^{(k)} = \mathfrak{g}^{(k+1)} = \mathbf{0}$ and thus $\mathfrak{g}$ is solvable.

Example 3.11. *The Heisenberg algebra is nilpotent:* Recall the Heisenberg algebra defined in Subsec. 3.1. For the basis $r_1, \dots r_n, p_1, \dots p_n, z$, we know that $[r_i, r_j] = [p_i, p_j] = [r_i, z] =$

$[p_i, z] = [z, z] = 0$ and $[r_i, p_j] = \delta_{ij} z$ for $1 \leq i, j \leq n$. It is easy to see that $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \{z\}$. and, since z is in the center, $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] = \mathbf{0}$. Thus, the Heisenberg algebra is nilpotent as well as solvable.

Example 3.12. *A solvable algebra that is not nilpotent:* Suppose $\mathfrak{g}$ is a two-dimensional, non-abelian algebra. Let x, y be the basis of $\mathfrak{g}$ with the commutation relation $[x, y] = x$. This algebra is solvable, for $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] = \{x\}$ and $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] = \mathbf{0}$. However, it is not nilpotent: $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] = \{x\}$, $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] = \{x\}$, and so on ad infinitum.

We can restate the definition of nilpotent in terms of the adjoint representation: For some n (depending only on $\mathfrak{g}$), $\operatorname{ad} x_1 \operatorname{ad} x_2 \ \ldots \operatorname{ad} x_n(y) = 0 \ \ \forall x_i, y \in \mathfrak{g}$. Another way to write this expression would be $[x_1, [x_{2,}[\ldots [x_n, y] \ldots] = 0$. This implies that $[x, [x, [\ldots, [x, y] \ldots] = 0$ if the product contains an n number of x's; in other words, if $(\operatorname{ad} x)^n = 0 \, \forall x \in \mathfrak{g}$, which is the definition of a nilpotent endomorphism. An element x of any Lie algebra $\mathfrak{g}$ is called **ad-nilpotent** if it is a nilpotent endomorphism. If $\mathfrak{g}$ is nilpotent, then all of its elements are ad-nilpotent. Furthermore, the converse is also true; this is Engel's Theorem and will be proved in the next subsection.

There are three further propositions concerning nilpotence that we will state. See Appendix A on page 18 if you are interested in the proofs.

Proposition 3.13.

i) Suppose $\mathfrak{g}$ is a nilpotent Lie Algebra. Then all subalgebras and homomorphic images of $\mathfrak{g}$ are also nilpotent.
ii) If $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, then $\mathfrak{g}$ is nilpotent as well.
iii) If $\mathfrak{g}$ is nilpotent and nonzero, $Z(\mathfrak{g}) \neq \mathbf{0}$.

3.4. **Engel's Theorem.** We now have the tools to prove our first big theorem, the converse of the property noted in the previous subsection connecting nilpotence and ad-nilpotence. Engel's Theorem gives a very nice result, for it allows one to show that a Lie algebra is nilpotent without directly calculating its descending central series. We will also use it to prove other theorems later in this paper and it can be applied in the representation theory of Lie algebras.

Theorem 3.14 (Engel). *If all elements of $\mathfrak{g}$ are ad-nilpotent, then $\mathfrak{g}$ is nilpotent as well.*

To prove this, we will first need two lemmas.

Lemma 3.15. *If $x \in \mathfrak{gl}(V)$ is a nilpotent endomorphism, then so too is $\operatorname{ad} x$.*

Proof. Let λ_x and ρ_x denote left and right multiplication by x (in the ring $\operatorname{End} V$), so $\lambda_x(y) = xy$ and $\rho_x(y) = yx$. We can write $\operatorname{ad} x = \lambda_x(y) - \rho_x(y)$. Note that λ_x and ρ_x commute:

$$\lambda_x \rho_x(y) = \lambda_x(yx) = x(yx) = (xy)x = \rho_x(xy) = \rho_x \lambda_x(y).$$

We know that x is nilpotent, so we can assume $x^n = 0$ for some $n \in \mathbb{N}$. Then we can use the binomial formula to write

$$(\operatorname{ad} x)^{2n-1} \quad = \quad \sum_{i=0}^{2n-1} (-1)^i \left(\begin{array}{c} 2n-1 \\ i \end{array} \right) \lambda_x(y)^{2n-1-i} \rho_x(y)^i$$

Each term in the sum is zero because $\lambda_x(y)^n = (xy)^n = x^n y^n = 0 \cdot y^n = 0$ and similarly for ρ_x; they are both nilpotent. Thus, $\operatorname{ad} x$ is nilpotent as well. ☐

Lemma 3.16. *Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$, with V a finite dimensional vector space. If $\mathfrak{g}$ consists of nilpotent endomorphisms and $V \neq \mathbf{0}$, then there exist nonzero $v \in V$ for which $x(v) = 0 \ \forall x \in \mathfrak{g}$.*

Proof. We will prove this by induction on the dimension of $\mathfrak{g}$. When $\dim \mathfrak{g} = 1$, $\mathfrak{g}$ is spanned by a single nilpotent linear transformation, which always has at least one eigenvector corresponding to the unique eigenvalue zero [4]. Thus, the assumption holds.

Now suppose $\dim \mathfrak{g} > 1$. First, we will show that $\mathfrak{g}$ contains an ideal of codimension one. To begin, take a maximal ideal $\mathfrak{h}$ of $\mathfrak{g}$ and consider the operation $\mathrm{ad}\, h = \psi(h) : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$, where $h \in \mathfrak{h}$ and $\psi(h)(x + \mathfrak{h}) = [h, x] + \mathfrak{h}$. It is easy to check that $\psi(h)$ is a Lie homomorphism:

$$\begin{aligned}
[\psi(h), \psi(f)](x + \mathfrak{h}) &= \psi(h)([f, x] + \mathfrak{h}) - \psi(f)([h, x] + \mathfrak{h}) \\
&= ([h, [f, x]] + \mathfrak{h}) - ([f, [h, x]] + \mathfrak{h}) \\
&= [h, [f, x]] - [f, [h, x]] + \mathfrak{h} \\
&= [[h, f], x] + \mathfrak{h} \\
&= \psi([h, f])(x + \mathfrak{h})
\end{aligned}$$

Note the use of the Jacobi Identity at step three. By the previous lemma, we can see that $\mathrm{ad}\, \mathfrak{h}$ must be nilpotent, as all $h \in \mathfrak{h}$ are nilpotent, and we can also notice that $\mathrm{ad}\, \mathfrak{h}$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$. Since $\dim \mathfrak{h} < \dim \mathfrak{g}$, we can apply the inductive hypothesis and conclude that there exists some non-zero element $y + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$ such that $\psi(h)(y + \mathfrak{h}) = 0$ for all $h \in \mathfrak{h}$. Written another way, $[h, y] \in \mathfrak{h} \ \forall h \in \mathfrak{h}$. Set $\mathfrak{h}' = \mathfrak{h} \oplus \mathrm{span}\, y$. This way, $\mathfrak{h}'$ is a Lie algebra in which $\mathfrak{h}$ is an ideal of codimension one. Now recall we chose $\mathfrak{h}$ to be a maximal subalgebra of $\mathfrak{g}$. Thus, $\mathfrak{g} = \mathfrak{h}'$, so $\mathfrak{h}$ is an ideal of codimension one in $\mathfrak{g}$.

Next, we will apply the induction hypothesis to $\mathfrak{h} \in \mathfrak{gl}(V)$. This means that there exists a nonzero vector v such that $x(v) = 0$ for all $x \in \mathfrak{h}$. Let W denote the subspace of all such vectors, $W = \{v \in V | x(v) = 0 \ \forall x \in \mathfrak{h}\}$. Let y be an element in $\mathfrak{g}$ that is not in $\mathfrak{h}$. Then all we need to show is that there exists a nonzero vector $w \in W$ such that $y(w) = 0$. For any vector $w \in W$ and $x \in \mathfrak{h}$, we have $xy(w) = yx(w) - [y, x](w)$. The first term on the right side is zero because we have assumed $x(w) = 0$ for any $x \in \mathfrak{h}$. Similarly, $[y, x] \in \mathfrak{h}$ so the second term is also zero. Thus, $xy(w) = 0$ and therefore $y(w) \in W$. This means that y carries the subspace W into itself; since y is nilpotent on V (recall that all elements of $\mathfrak{g}$ are nilpotent) there must be a nonzero vector $v \in V$ such that $y(v) = 0$. Thus, a v exists for which $\mathfrak{g}(v) = 0$. □

Note. This proof draws from the proofs given in [4], [6], and [8].

In essence, Lemma 3.16 shows that there is a common eigenvector for a Lie algebra consisting of nilpotent endomorphisms [4]. With these two lemmas, we now we have the means with which to prove Engel's Theorem.

Proof of Engel's Theorem. The initial conditions give us a Lie algebra $\mathfrak{g}$ whose elements are all ad-nilpotent. Thus, we can apply Lem. 3.16 to the algebra $\mathrm{ad}\, \mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ and conclude that there exists $x \in \mathfrak{g}$ for which $[y, x] = 0 \ \forall y \in \mathfrak{g}$. In other words, $Z(\mathfrak{g}) \neq \mathbf{0}$. Next, consider $\mathfrak{g}/Z(\mathfrak{g})$. We will induct on $\dim \mathfrak{g}$ to show that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent. When $\dim \mathfrak{g} = 0$, $\mathfrak{g}/Z(\mathfrak{g})$ is clearly nilpotent. Now suppose $\dim \mathfrak{g} > 0$. Note that $\mathfrak{g}/Z(\mathfrak{g})$ has a dimension less than $\mathfrak{g}$ and consists of ad-nilpotent elements. By the induction hypothesis, $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent and then applying Prop. 3.13 shows us that $\mathfrak{g}$ is nilpotent. □

Example 3.17. *Nilpotence of a Heisenberg algebra*: We have already shown the nilpotence of a Heisenberg algebra in Ex. 3.8 from the previous subsection. Now we will use Engel's Theorem. Consider a Heisenberg algebra for which $n = 2$. The basis becomes r_1, r_2, p_1, p_2, z with the commutation relations $[r_i, r_j] = [p_i, p_j] = [r_i, z] = [p_i, z] = [z, z] = 0$ and $[r_i, p_j] = \delta_{ij} z$ for

$1 \leq i, j \leq 2$. With this, we find that the adjoint representations are

$$\operatorname{ad} r_1 = \begin{pmatrix} & & \mathbf{0} & & \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad} r_2 = \begin{pmatrix} & & \mathbf{0} & & \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \operatorname{ad} p_1 = \begin{pmatrix} & & \mathbf{0} & & \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\operatorname{ad} p_2 = \begin{pmatrix} & & \mathbf{0} & & \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad \operatorname{ad} z = \begin{pmatrix} \mathbf{0} \end{pmatrix}$$

Clearly, $\operatorname{ad} z$ is ad-nilpotent, and a quick calculation shows that $(\operatorname{ad} r_i)^2 = (\operatorname{ad} p_i)^2 = 0$ for $i = 1, 2$, so the other elements are ad-nilpotent as well. Thus, by Engel's Theorem, this Heisenberg algebra is nilpotent. In this particular case, it was easier to determine the nilpotence using the descending central series, but that was only because this example is so simplistic. In many circumstances, it is much easier to determine if the the adjoint representations of the elements in a Lie algebra are ad-nilpotent.

An equivalent version of Engel's Theorem gives another useful way of finding nilpotent algebras. It is stated here as a corollary.

Corollary. *Under the hypothesis of Engel's Theorem, there is a basis in V in which every element of $\mathfrak{g}$ is represented by a strictly upper triangular matrix (that is, a matrix with only zeros on and below the diagonal).*

4. SEMISIMPLE LIE ALGEBRAS

In the previous section, we showed that a Lie algebra can be written as a direct sum of its solvable radical and a semisimple algebra, among many other concepts. It is therefore worthwhile to study semisimple Lie algebras in more detail. In this section, we will prove another fundamental theorem and develop two powerful criteria which quickly determine the solvability and semisimplicity of an algebra. Finally, we will show that a semisimple Lie algebra can be written as a direct sum of its simple ideals. *It is important to note that we will assume that $\mathbb{F}$ has a characteristic of zero and is algebraically closed for the rest of this paper.*

Recall that a Lie algebra is semisimple if it contains no nonzero solvable ideals and simple if it contains no proper ideals at all (and is not abelian). There is also second definition of semisimple that will become useful in this section: we call $x \in \operatorname{End} V$ **semisimple** if the roots of its minimal polynomial over $\mathbb{F}$ are all distinct. When $\mathbb{F}$ is assumed to be algebraically closed as we are doing, this is equivalent to x being diagonalizable [4]. With this definition, we can prove a lemma which will come in handy in Subsec. 4.2.

Lemma 4.1. *If x is semisimple, then so is $\operatorname{ad} x$.*

Proof. Choose a basis $v_1, \ldots, v_n$ of V so that x has the matrix $\operatorname{diag}(a_1, \ldots, a_n)$ and $\mathfrak{gl}(V)$ has the standard basis $\{e_{ij}\}$ relative to it. Using the equations $e_{ij}(v_k) = \delta_{jk} v_i$ and $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kl}$, which can be simply worked out, we can quickly find that $\operatorname{ad} x\ (e_{ij}) = (a_i - a_j) e_{ij}$. Thus, $\operatorname{ad} x$ has a diagonal matrix, which means it is semisimple. □

Now, on to the theorem.

4.1. Lie's Theorem.

This theorem gives us useful information about solvable Lie algebras. It tells us that a vector space V contains a common eigenvector for endomorphisms in $\mathfrak{g}$, a fact we will use to prove another important theorem later in this paper.

Theorem 4.2 (**Lie**). *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable Lie algebra, with V finite dimensional. If $V \neq \mathbf{0}$, there exists a nonzero vector $v \in V$ that is an eigenvalue of x for all $x \in \mathfrak{g}$.*

Proof. The proof proceeds in a similar fashion as Lem. 3.16, the second lemma we proved for Engel's Theorem. First, we want to show that $\mathfrak{g}$ contains an ideal of codimension one. This is simpler than last time: since $\mathfrak{g}$ is solvable, $[\mathfrak{g},\mathfrak{g}] \neq \mathfrak{g}$, so $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is a non-zero abelian algebra and any subspace of $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ will be ideal. Furthermore, the inverse image (in $\mathfrak{g}$) of any subspace with codimension one in $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ will be an ideal with codimension one in $\mathfrak{g}$. Call this ideal $\mathfrak{h}$.

Next, we will use induction on $\dim \mathfrak{g}$ to see that common eigenvectors exist for $\mathfrak{h}$. When $\mathfrak{h} = \mathbf{0}$, $\mathfrak{g}$ is abelian and $\dim \mathfrak{g} = 1$. In this situation, $\mathfrak{g}$ contains a common eigenvector; any eigenvector that is a basis vector of $\mathfrak{g}$ will do. Now suppose $\dim \mathfrak{g} > 1$. By the induction hypothesis, we can assume that there is a vector $v_0 \in V$ that is an eigenvector $\forall\, x \in \mathfrak{h}$ because $\dim \mathfrak{h} < \dim \mathfrak{g}$. Denote the eigenvalue of x corresponding to v_0 as $\lambda(x)$ and consider the subspace $W = \{v \in V | x(v) = \lambda(x)v \ \ \forall x \in \mathfrak{h}\}$. Clearly, $W \neq \mathbf{0}$.

Now we will show that W is invariant under all of $\mathfrak{g}$. Let y be an element in $\mathfrak{g}$ that is not in $\mathfrak{h}$. We want to show that y carries some vector $w \in W$ into a multiple of itself, and it is enough to show that y carries W into itself. In this interest, consider a nonzero $w \in W$ and $x \in \mathfrak{h}$ and recall that $[x, y] \in \mathfrak{h}$ as well. We can write

$$\begin{aligned} x(y(w)) &= y(x(w)) + [x,y](w) \\ &= \lambda(x)y(w) + \lambda([x,y])w \end{aligned} \tag{4.1}$$

We want $y(w) \in W$; this will happen iff $\lambda([x, y]) = 0$ for all $x \in \mathfrak{h}$.

Thus, we must prove that $\lambda([x, y]) = 0$. To do this, let us consider U, defined as the subspace of V spanned by $w,\ y(w),\ y^2(w), \dots,\ y^i(w), \dots$. Evidently, the subspace is preserved by y; we will use induction to show that U is preserved by x as well. It is clear that $\mathfrak{h}$ carries w into a multiple of itself and (4.1) shows that $\mathfrak{h}$ carries $y(w)$ into a linear combination of w and $y(w)$. Hence, $\mathfrak{h}$ carries w and $y(w)$ into U. In general, for any $x \in \mathfrak{h}$ and $k > 1$,

$$x(y^k(w)) = y(x(y^{k-1}(w))) + [x,y](y^{k-1}(w)) \tag{4.2}$$

By the induction hypothesis, $x(y^{k-1}(w) \in U$ and $[x,y](y^{k-1}(w)) \in U$ because $x, [x,y] \in \mathfrak{h}$. Thus, $\mathfrak{h}$ carries $y^k(w)$ into U. Now consider: in terms of the basis for U, the action of $x \in \mathfrak{h}$ is upper triangular with $\lambda(x)$ on the diagonal entries. Suppose $\dim U = n$. Then the trace of the restriction of x to U, $\mathrm{Tr}_u(x)$, is $n\lambda(x)$. This is also true for the elements of $\mathfrak{h}$ of the form $[x, y]$. However, $[x, y]$ is he commutator of two endomorphisms of U, so its trace is 0. We can therefore reason that $n\lambda([x, y]) = 0$ and, since we have assumed $\mathrm{char}\, \mathbb{F} = 0$, this implies $\lambda([x, y]) = 0$. This ultimately means that $\mathfrak{g}$ leaves W invariant.

Finally, note that we can write any $x \in \mathfrak{g}$ as $x = \alpha + \beta z$ where $\alpha \in \mathfrak{h}$ and $\beta \in \mathbb{F}$. Because $\mathbb{F}$ is algebraically closed, we can find an eigenvector $v_0 \in W$ of z. We are at the end, for v_0 is a common eigenvector for $\mathfrak{g}$. ☐

Note. This proof draws from those given in [4], [6], and [8].

We can immediately derive a useful corollary from Lie's Theorem, analogous to the corollary from Engel's Theorem.

Corollary. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a solvable Lie algebra, with V finite dimensional as always. Then there is a basis in V in which every element of $\mathfrak{g}$ is represented by an upper triangular matrix.*

In the previous proof, we assumed that $\mathrm{char}\, \mathbb{F} = 0$. Lie's theorem fails when $\mathbb{F}$ has prime characteristic. For an example of this, see Appendix B on page 19.

4.2. **Cartan's Criteria.** Now we are to a point where we can develop a simple way of determining the solvability and semisimplicity of a Lie algebra. Knowing this information will aid us in its decomposition. In this subsection, we will define two criteria, one which examines solvability and the other semisimplicity. First, however, we will introduce a new form.

4.2.1. *The Killing Form.* Using the adjoint representation, we can construct a symmetric bilinear form κ called the **Killing form**, defined as $\kappa(x,y) = \mathrm{Tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$ for $x, y \in \mathfrak{g}$. Note that κ is associative in that $\kappa([x,y],z) = \kappa(x,[y,z])$. Additionally, the Killing form is **nondegenerate** when its radical $\mathfrak{r} = \{x \in \mathfrak{g} | \kappa(x,y) = 0\ \forall y \in \mathfrak{g}\}$ is zero. This radical is more than just a subspace: $\kappa([x,y],z) = -\kappa([y,x],z) = -\kappa(y,[x,z]) = 0$ for any $x \in \mathfrak{g}$, $y \in \mathfrak{r}$, and $z \in \mathfrak{g}$, which means $[x,y] \in \mathfrak{r}$. In other words, the radical is an *ideal* of $\mathfrak{g}$. We will see in the following subsections that a nondegenerate Killing form can tell us helpful information. This is useful because we have a simple way of testing nondegeneracy from linear algebra. Fix a basis $x_1 \dots, x_n$ of $\mathfrak{g}$; κ is nondegenerate iff the $n \times n$ matrix with $\kappa(x_i, x_j)$ as its (i,j) entry has nonzero determinate [4].

Example 4.3. *The Killing form of* $\mathfrak{su}(2)$*:* Recall from the previous examples throughout this paper that, using the basis a_1, a_2, a_3, the adjoint representations are:

$$\mathrm{ad}\, a_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathrm{ad}\, a_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}\, a_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now we can calculate the 3×3 matrix described above. For example,

$$\kappa(a_1, a_1) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -2 \text{ and } \kappa(a_1, a_2) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0.$$

One can find that $\kappa(a_i, a_j) = -2\delta_{ij}$. Using this expression, we get the matrix

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} \text{ for } \kappa.$$

It has a determinant of $(-2)^3 = -8$, so κ is nondegenerate.

4.2.2. *Cartan's First Criterion.* This criterion uses the Killing form to determine the solvability of a Lie algebra.

Theorem 4.4 (**Cartan's First Criterion**). *A Lie algebra* $\mathfrak{g}$ *is solvable if and only if* $\kappa(x,y) = 0$ *for all* $x \in [\mathfrak{g}, \mathfrak{g}],\ y \in \mathfrak{g}$.

We will begin with a proposition.

Proposition 4.5. *Let* $\mathfrak{g} \subset \mathfrak{gl}(V)$ *(V finite dimensional) be a Lie algebra with the property* $\mathrm{Tr}(xy) = 0$ *for all* $x, y \in \mathfrak{g}$. *Then* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*

Proof. In this proof, we will use *Jordan decomposition* (see page 17 of [4]), defined as follows: for any $x \in [\mathfrak{g}, \mathfrak{g}]$, we have $x = s + n$, where s is the semisimple part and n is the nilpotent part of x. It is known that s and n are unique and that they commute [4]. To show that x is nilpotent, we want to show that $s = 0$. Fix a basis in V relative to which s is a matrix with the eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$ on the diagonal and zeros everywhere else. We will consider all operators on V with respect to this basis and take the standard basis $\{e_{ij}\}$ for $\mathfrak{gl}(V)$. Set $\overline{s}$ as the complex conjugate of s; then $\overline{s} = \mathrm{diag}(\overline{\lambda}_1, \overline{\lambda}_2, \dots, \overline{\lambda}_n)$. We can write $\overline{s}$ as a polynomial in s using Lagrange interpolation; the assigned values are certain because $\lambda_i = \lambda_j$ implies $\overline{\lambda}_i = \overline{\lambda}_j$ and thus there is a polynomial $p(x)$ with $p(\lambda_i) = \overline{\lambda}_i$.

Next, consider ad of $\mathfrak{gl}(V)$ restricted to $\mathfrak{g}$. We know $\mathrm{ad}\, x = \mathrm{ad}\, s + \mathrm{ad}\, n$, and it is simple to show that it is a Jordan decomposition: $\mathrm{ad}\, s$ is semisimple by Lem. 4.1, $\mathrm{ad}\, n$ is nilpotent by Lem. 3.15, and they commute, for $[\mathrm{ad}\, s, \mathrm{ad}\, n] = \mathrm{ad}\, [s, n] = 0$. Note that $\mathrm{ad}\, s$ has eigenvalues $\lambda_i - \lambda_j$ on the basis $\{e_{ij}\}$. Due to the fact that $\mathrm{ad}\, s + \mathrm{ad}\, n$ is a Jordan composition of $\mathrm{ad}\, x$, we know that $\mathrm{ad}\, s$ is a polynomial in $\mathrm{ad}\, x$ [4]. Furthermore, $\mathrm{ad}\, \overline{s}$ is also diagonal (with eigenvalues $\overline{\lambda}_i - \overline{\lambda}_j$) and a polynomial in $\mathrm{ad}\, s$; this means it is a polynomial in $\mathrm{ad}\, x$ as well. This fact implies $[\overline{s}, y] \in \mathfrak{g}$ for all $y \in \mathfrak{g}$.

Putting that find on hold for a paragraph, let us go back to the polynomial $p(x)$. Since $\overline{s} = p(s)$, we can see that $\overline{s}$ and n commute. Thus, the product $\overline{s}n$ is nilpotent and has trace zero. (Recall that nilpotent endomorphisms are strictly upper triangular). This allows us to see that $\mathrm{Tr}(\overline{s}x) = \mathrm{Tr}(\overline{s}n) + \mathrm{Tr}(\overline{s}s) = \mathrm{Tr}(\overline{s}s) = \sum \overline{\lambda}_i \lambda_i$.

Alternatively, we can write $x = \sum [a_i, b_i]$ with $a_i, b_i \in \mathfrak{g}$ because $x \in [\mathfrak{g}, \mathfrak{g}]$. Then for each term, $\mathrm{Tr}(\overline{s}x) = \mathrm{Tr}(\overline{s}[a, b]) = \mathrm{Tr}(\overline{s}ab) - \mathrm{Tr}(\overline{s}ba) = \mathrm{Tr}(\overline{s}ab) - \mathrm{Tr}(a\overline{s}b) = \mathrm{Tr}([\overline{s}, a]b)$. We found in the paragraph before last that $[\overline{s}, a] \in \mathfrak{g}$, so $\mathrm{Tr}([\overline{s}, a]b) = 0$ by the proposition assumptions. This means $\sum \overline{\lambda}_i \lambda_i = 0$, which forces all λ_i to be zero; thus $s = 0$ and x must be nilpotent. All we need to do now is apply Engel's Theorem (3.14), and we see that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

Note. This proof draws from those given in [9], [8], and [10].

Now we have the tools to proceed to Cartan's first criterion.

Proof of Cartan's First Criterion. Recall that the theorem is an 'if and only if' statement.

First, assume $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = 0$; we will show that $\mathfrak{g}$ is solvable. Consider the representation ad of $\mathfrak{g}$ on $\mathfrak{g}$. It is clear that $\mathrm{ad}\, \mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(\mathfrak{g})$. Apply the above proposition to find that $[\mathrm{ad}\, \mathfrak{g}, \mathrm{ad}\, \mathfrak{g}]$ is nilpotent and, recalling that a nilpotent derived algebra implies a solvable algebra (Subsec. 3.3.2), we see that $\mathrm{ad}\, \mathfrak{g}$ is solvable. Finally, since $\ker \mathrm{ad} = Z(\mathfrak{g})$ is solvable, $\mathfrak{g}$ is solvable as well (Subsec. 3.3.1)

Conversely, let us now assume $\mathfrak{g}$ is solvable and show that implies $\kappa(x, y) = 0$. By applying Lie's Theorem (4.2) to $\mathrm{ad}\, \mathfrak{g}$, we see that the matrices for $\mathrm{ad}\, x$ are triangular. For $x \in [\mathfrak{g}, \mathfrak{g}]$, the diagonal elements of $\mathrm{ad}\, x$ are zero: consider any $\mathrm{ad}\, a \cdot \mathrm{ad}\, b - \mathrm{ad}\, b \cdot \mathrm{ad}\, a$. Thus, $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = \mathrm{Tr}(0 \cdot \mathrm{ad}\, y) = 0$. □

Example 4.6. *The Heisenberg Algebra:* We have already found that the Heisenberg algebra was solvable in Ex. 3.8 on page 7. Now we will show this by applying Cartan's first criterion instead. Let $\mathfrak{g}$ denote the Heisenberg algebra and recall that it has the basis $r_1, \dots r_n, p_1, \dots p_n, z$ with the commutation relations $[r_i, r_j] = [p_i, p_j] = [r_i, z] = [p_i, z] = [z, z] = 0$ and $[r_i, p_j] = \delta_{ij} z$ for $1 \leq i, j \leq n$. To demonstrate that $\mathfrak{g}$ is solvable, we must show that $\kappa(x, y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}],\ y \in \mathfrak{g}$. First, note that $[\mathfrak{g}, \mathfrak{g}] = \{z\}$. It is easily seen that $\mathrm{ad}\, z = (\mathbf{0})$. Multiplying this matrix by the adjoint representation of any other element in $\mathfrak{g}$ will also result in the zero matrix, so $\kappa(x, y) = \mathrm{Tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}],\ y \in \mathfrak{g}$ and therefore $\mathfrak{g}$ is solvable by Cartan's first criterion.

4.2.3. *Cartan's Second Criterion.* This criterion uses the Killing form to determine if a Lie algebra is semisimple. Recall that a nondegenerate κ means that its radical $\mathfrak{r} = 0$, where $\mathfrak{r} = \{x \in \mathfrak{g} | \kappa(x, y) = 0\ \forall y \in \mathfrak{g}\}$.

Theorem 4.7 (**Cartan's Second Criterion**)**.** *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is nondegenerate.*

Proof. First, let us suppose that $\mathfrak{g}$ is semisimple, so $\text{rad}\ \mathfrak{g} = \mathbf{0}$. We will show κ is nondegenerate. Let $\mathfrak{r}$ be the radical of κ. By definition, $\text{Tr}(\text{ad}\ x\ \text{ad}\ y) = 0$ for all $x \in \mathfrak{r}, y \in \mathfrak{g}$. Cartan's First Criterion tells us that $\text{ad}\ \mathfrak{r}$ restricted to $\mathfrak{g}$ is solvable, so $\mathfrak{r}$ is solvable. Furthermore, recall that the radical is an ideal of $\mathfrak{g}$ (Subsec. 4.2.1). Note that requiring $\text{rad}\ \mathfrak{g} = \mathbf{0}$ indicates that any nonzero abelian ideals of $\mathfrak{g}$ would be inside $\text{rad}\ \mathfrak{g}$ [4]. Thus, $\mathfrak{r} \subset \text{rad}\ \mathfrak{g} = \mathbf{0}$ and κ is nondegenerate.

Now, let us go the other direction. Suppose κ is nondegenerate: let $\mathfrak{r} = 0$. We will show that every abelian ideal $\mathfrak{q}$ of $\mathfrak{g}$ is included in $\mathfrak{r}$; this means $\text{rad}\ \mathfrak{g} = 0$ and thus $\mathfrak{g}$ is semisimple. Suppose $x \in \mathfrak{q}$ and $y \in \mathfrak{g}$ and consider the inside of the Killing form, $\text{ad}\ x\ \text{ad}\ y$. With x and y as they are, $\text{ad}\ x\ \text{ad}\ y$ maps $\mathfrak{g} \to \mathfrak{g} \to \mathfrak{q}$ and $(\text{ad}\ x\ \text{ad}\ y)^2$ maps $\mathfrak{g} \to [\mathfrak{g}, \mathfrak{g}] \to [\mathfrak{q}, \mathfrak{q}] = \mathbf{0}$ (because $\mathfrak{q}$ is abelian). Thus, $\text{ad}\ x\ \text{ad}\ y$ is nilpotent; this indicates that $0 = \text{Tr}(\text{ad}\ x\ \text{ad}\ y) = \kappa(x, y)$ and so $\mathfrak{q} \subset \mathfrak{r} = \mathbf{0}$. □

Note. This proof draws from that given in [4] and [10].

Example 4.8. $\mathfrak{su}(2)$ *is semisimple*: We showed in Ex. 3.5 that $\mathfrak{su}(2)$ is simple, and thus it is semisimple. However, we now have a more manageable way of determining this. Recall from Ex. 4.3 that the Killing form of $\mathfrak{su}(2)$ is nondegenerate. By Cartan's Second Criterion, it is therefore semisimple. While calculating the determinate of the Killing form of a Lie algebra can be labor-intensive, requiring one to work with n^2 variables, note that it is still much easier than any other method [8].

4.3. Simple Ideals of $\mathfrak{g}$.

We have come finally to the goal of this paper. We have the necessary means to show that a semisimple Lie algebra can be decomposed into simple ideals. Thus, we can show that one can write a Lie algebra as a direct sum of a solvable radical and simple ideals. The analysis of the Lie algebra's structure has been reduced to studying simple and solvable Lie algebras!

Theorem 4.9. *Let* $\mathfrak{g}$ *be semisimple. Then there exist simple ideals* $\mathfrak{g}, \dots, \mathfrak{g}_t$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{g} \oplus \dots \oplus \mathfrak{g}_t$. *Every simple ideal of* $\mathfrak{g}$ *coincides with one of the* $\mathfrak{g}_i$.

Proof. Let $\mathfrak{a}$ be any nonzero ideal of $\mathfrak{g}$, and consider $\mathfrak{a}^\perp = \{x \in \mathfrak{g} | \kappa(x, y) = 0 \ \ \forall y \in \mathfrak{a}\}$. Due to the associativity of κ (Subsec. 4.2.1), $\mathfrak{a}^\perp$ is also an ideal, and thus so too is $\mathfrak{a} \cap \mathfrak{a}^\perp$. We apply Cartan's First Criterion (4.4) find that $\mathfrak{a} \cap \mathfrak{a}^\perp$ is solvable; since $\mathfrak{g}$ is semisimple, it contains no solvable ideals and thus $\mathfrak{a} \cap \mathfrak{a}^\perp = \mathbf{0}$. Furthermore, we know $\dim \mathfrak{a} + \dim \mathfrak{a}^\perp = \dim \mathfrak{g}$, which implies that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$. Clearly, $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are semisimple because any ideal of $\mathfrak{a}$ or $\mathfrak{a}^\perp$ would also be an ideal of $\mathfrak{g}$, and $\mathfrak{g}$ cannot have solvable ideals.

Now we will use induction on $\dim \mathfrak{g}$ to decompose $\mathfrak{g}$ into simple ideals. If $\mathfrak{g}$ does not include a nonzero proper ideal, $\mathfrak{g}$ is simple already and no further work is necessary. If that is not the case, then suppose $\mathfrak{g}_1$ is a minimal nonzero ideal; we can write $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_1^\perp$. As noted above, $\mathfrak{g}_1$ and $\mathfrak{g}_1^\perp$ are both semisimple, and by minimality, $\mathfrak{g}_1$ is also simple. Then, we can apply the induction hypothesis to split $\mathfrak{g}_1^\perp$ into a direct sum of simple ideals, which are also ideals of $\mathfrak{g}$. Hence, $\mathfrak{g}$ can be decomposed.

Lastly, we want to show that these simple ideals are unique. Suppose $\mathfrak{q}$ is any ideal of $\mathfrak{g}$; $[\mathfrak{q}, \mathfrak{g}]$ is then also an ideal. Since $Z(\mathfrak{g}) = \mathbf{0}$ by $\mathfrak{g}$'s semisimplicity, we must have $[\mathfrak{q}, \mathfrak{g}] = \mathfrak{q}$. However, we can also write $[\mathfrak{q}, \mathfrak{g}] = [\mathfrak{q}, \mathfrak{g}_1] \oplus \dots \oplus [\mathfrak{q}, \mathfrak{g}_t]$. For both expression to be true, all but one summand must be zero. Let $[\mathfrak{q}, \mathfrak{g}_i] = \mathfrak{q}$. Then $\mathfrak{q} \subset \mathfrak{g}_i$ and, due to the fact that $\mathfrak{g}_i$ is simple, $\mathfrak{g}_i = \mathfrak{q}$. Hence, every simple ideal matches one of the $\mathfrak{g}_i$. □

Note. This proof draws from those given in [4] and [9].

Example 4.10. *The decomposition of* $\mathfrak{so}(4)$*:* We have encountered the linear Lie algebra $\mathfrak{so}(4)$ before, in Ex. 3.10 on page 7. We used the basis $a_1, a_2, a_3, b_1, b_2, b_3$ and we found that the commutation relations are $[a_i, a_j] = -\epsilon_{ijk}a_k$, $[b_i, b_j] = -\epsilon_{ijk}b_k$ and $[a_i, b_j] = 0$ for $i, j = 1, 2, 3$. We also showed that $\mathfrak{so}(4)$ is semisimple.

Now consider the subgroups $\{a_i\}$ and $\{b_i\}$. They are in fact Lie algebras: $\{a_i\}$ has the commutation relations $[a_i, a_j] = -\epsilon_{ijk}a_k$ and $\{b_i\}$ has $[b_i, b_j] = -\epsilon_{ijk}b_k$. These relations should look familiar: they are the same as those of $\mathfrak{su}(2)$, one of the Lie algebras we have been using as an example throughout this paper. Since the relations are the same, we can see that $\{a_i\}$ and $\{b_i\}$ are both isomorphic to $\mathfrak{su}(2)$ in a manner similar to that used in Ex. 3.6. Finally, we conclude from the fact that $\mathfrak{so}(4) = a_i \oplus b_i$ that $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Recall further that we have shown $\mathfrak{su}(2)$ to be simple (Ex. 3.5). Thus, $\mathfrak{so}(4)$ is an example of how a semisimple Lie algebra can be written as a direct sum of simple Lie algebras.

5. MORE INFORMATION

We have achieved our goal of showing that Lie algebras can be decomposed into a direct sum of solvable and simple ideals, but this is not always immediately useful. Representation theory arises as the method to use to study these ideals further. Due to length constraints, this paper will not discuss this important subject in the detail of the previous sections, but a brief summary of some basic ideas will be presented. We will examine how to study the structure of simple Lie algebras through *roots* and *weights*.

5.1. Roots.

To understand roots, we will first consider a special type of Lie subalgebra. A **Cartan subalgebra (CSA)** is a nilpotent subalgebra $\mathfrak{h}$ (of a Lie algebra $\mathfrak{g}$) that is self-normalizing, i.e., $\mathfrak{h} = \{x \in \mathfrak{g} | [x, \mathfrak{h}] \subset \mathfrak{h}\}$. Every Lie algebra has at least one CSA, and all CSAs have the same dimension [2]. When $\mathfrak{g}$ is semisimple, a CSA is a maximal commutative subalgebra (Note however, that a maximal commutative subalgebra is not necessarily a CSA) [9]. In the semisimple situation, $\mathfrak{h}$ is also abelian [4]. For this subsection, we will assume $\mathfrak{g}$ is a semisimple Lie algebra over the complex numbers. Let $\mathfrak{h}^*$ denote the dual space of $\mathfrak{h}$; it is the set of all linear forms that map $\mathfrak{h}$ onto the set of all complex numbers [2]. Due to the fact that $\mathfrak{h}$ is nilpotent, all $h \in \mathfrak{h}$ are ad-nilpotent, and thus $\operatorname{ad} h$ is diagonalizable. Furthermore, $\mathfrak{h}$ is abelian and commuting linear maps can have common eigenvalues, so we can expect to find $x \in \mathfrak{g}$ which are simultaneous eigenvalues of $\operatorname{ad} h$ on $\mathfrak{g}$.

Given a nonzero $x \in \mathfrak{g}$, we define a **root** as a linear form $\alpha \in \mathfrak{h}^*$ such that $[h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}$. For each root, we can find the associated **root space**, $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x \; \forall h \in \mathfrak{h}\}$. Using this, we can decompose $\mathfrak{g}$ into a direct sum of the CSA and nonzero root spaces, which is called the *root space decomposition*:

$$\mathfrak{g} \;=\; \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \quad \text{where } \Phi = \{\alpha \in \mathfrak{h}^* | \alpha \neq 0,\ \mathfrak{g}_\alpha \neq \mathbf{0}\}.$$

Note that there are several ways to find such a set of roots. We call Φ a *root system* of a Euclidean space; the study and classification of such systems is useful to many areas of mathematics, but beyond the scope of this paper. Here, the important idea is that Φ characterizes $\mathfrak{g}$ completely [4].

We can define a partial lexicographic ordering on a set of roots by considering the subspace $\mathfrak{h}^*$ spanned by real linear combinations of roots, denoted $\mathfrak{h}^*_{\mathbb{R}}$. Suppose $\gamma_1, \ldots, \gamma_l$ is an ordered basis for $\mathfrak{h}^*_{\mathbb{R}}$. Then $\xi \in \mathfrak{h}^*_{\mathbb{R}}$ can be written as $\xi = r_1\gamma_1 + \ldots r_l\gamma_l$, where $r_i \in \mathbb{R}$. We call $\xi > 0$ if the first nonzero component of ξ is positive, and we say that $\xi > \eta$ when $\xi - \eta > 0$. We define a root α as a *positive root* if $\alpha > 0$, and we call a positive root a *simple root* if it is not the sum of two positive roots. The simple roots form a basis for $\mathfrak{h}^*$ [2]. For each Φ, it is possible to find a unique set of simple roots such that each positive root $\alpha \in \Phi$ can be written as a combination of simple roots with all non-negative (or all non-positive) coefficients [4]. With such a set of simple roots, it is possible to classify irreducible root systems, which in turn determine the structure of simple Lie algebras (up to isomorphism). There are a finite number of irreducible root systems, so it is possible to determine all possible simple Lie algebras [2]. Most of these fall into one of four infinite families, called the *classical Lie algebras* A_l, B_l, C_l, and D_l, which can be simply described using matrices. There are also five more algebras called "exceptional". It can be shown that these simple Lie algebras are the building blocks of semisimple Lie algebras [5]. Thus, roots and root systems are a powerful tool for studying semisimple Lie algebras.

5.2. **Weights.** *Weights* are also very useful in the classification of representations. They are a broader version of roots. In fact, a root is just a weight specialized to the adjoint representation [11].

First, let us suppose that $\mathfrak{g}$ is semisimple. Let $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ on a (complex) finite-dimensional vector space V and consider $h \in \mathfrak{h}$, where $\mathfrak{h}$ is once again a CSA. Define $v \in V$ as a **weight vector** when v is a simultaneous eigenvector of every $\phi(h)$. A *weight* of v is the corresponding eigenvalue for $\phi(h)$; it is a linear map $\lambda : \mathfrak{h} \to \mathbb{C}$ and thus $\lambda \in \mathfrak{h}^*$, the dual space of $\mathfrak{h}$. For each λ, there is an associated **weight space** $V_\lambda = \{v \in V |\ hv = \lambda(h)v \ \ \forall h \in \mathfrak{h}\}$. Note that there may not be any weight vectors with a particular λ as a weight or written another way, $V_\lambda = \mathbf{0}$. We call λ a **weight** (of ϕ) when $V_\lambda \neq \mathbf{0}$. When $\mathfrak{g}$ is semisimple then V can be written as a direct sum of its weight spaces: $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ [11].

Weights have some very useful and important properties. Two relevant to this brief overview will be listed here. First, V is spanned by weight vectors, and there is a finite number of weights. Furthermore, the weights are integral forms, which means that all values $\lambda(h\alpha)$ (for α, a root in the root system) are integers [9]. Now, we consider a lemma essential to utilizing weights.

Lemma 5.1. *Let v, λ, and V_λ be defined as above; let α be any root; and let x_α be the corresponding root element. Then x_α maps V_λ into $V_{\lambda+\alpha}$. Stated otherwise, if $x_\alpha v$ is not zero, then $x_\alpha v = (\lambda + \alpha)v$, that is, $x_\alpha v$ is a weight vector of ϕ with weight $(\lambda + \alpha)$.*

Proof. See Samelson [9], pg. 95. □

Using the partial ordering defined with respect to roots in the above section, we can use lemma 5.1 to create a ladder of weights. The **highest weight** is a weight λ such that $(\lambda + \alpha)$ is not a weight of ϕ for any positive root α. It can be shown that if ϕ is an irreducible representation, then there exists exactly one highest weight; all other weights are of the form $\lambda - \sum n_i \alpha_i$, n_i non-negative integers. The converse is also true. Thus, the highest weight *determines* the representation; if two irreducible representations of $\mathfrak{g}$ have the same highest weight, they are equivalent [9]. Therefore, to classify irreducible representations, one can find all possible highest weights.

It is also possible to consider weights when $\mathfrak{g}$ is not necessarily semisimple by using the the concept of modules [11]. Studying a module is equivalent to inspecting the algebra's representations, it is merely a matter of emphasis: representations emphasize homomorphisms, modules emphasize the vector spaces acted upon by the linear transformations mapped to by the homomorphism. Sophus Lie proved that any module over a solvable Lie algebra has a weight λ and that, for a nilpotent Lie algebra, one can find a basis for any module consisting of weight vectors [11]. These theorems were put into practice earlier in this subsection when we used the CSA.

The use of weights and roots in the study of Lie algebras makes it easier to classify the irreducible representations of a Lie algebra, which provide information about the structure of the algebra in question.

6. CONCLUSIONS

The study of Lie algebras can be a powerful tool in mathematics and physics. In this paper, we used Engel's Theorem, Lie's Theorem, and other ideas to find that a Lie algebra can be written as a direct sum of the maximal solvable ideal and a semisimple algebra, which can in turn be decomposed further into simple ideals. Thus, the study of the structure of a Lie algebra can be mostly reduced to the study of solvable and simple Lie algebras. We proved Cartan's Criteria, giving us an easy way to determine if a Lie algebra is solvable or semisimple; if it is semisimple, then its structure can be completely characterized when one can determine the structure of the simple ideals of which it is a direct sum. We have the means to find the structure of the simple ideals through roots and weights. Thus, we have outlined in part a method to determine the structure of a Lie algebra, which characterizes completely the local structure of the Lie group associated with it. Given the prevalence and usefulness of Lie groups in mathematics and physics, Lie algebras are thus useful indeed.

APPENDIX A

Solvability propositions and proofs: (From Sec. 3.3.1 on page 7)

Proposition. *Suppose* $\mathfrak{g}$ *is a solvable Lie Algebra. Then all subalgebras and homomorphic images of* $\mathfrak{g}$ *are also solvable.*

Proof. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{h}^{(i)} \subset \mathfrak{g}^{(i)}$. Thus, if $\mathfrak{g}^{(n)} = \mathbf{0}$ for some n, then $\mathfrak{h}^{(n)} = \mathbf{0}$ as well. Now consider an epimorphism $\phi : \mathfrak{g} \to \mathfrak{h}$. We will show that $\phi(\mathfrak{g}^{(i)}) = \mathfrak{h}^{(i)}$ through proof by induction. By the definition of an epimorphism, we know that $\phi(\mathfrak{g}) = \mathfrak{h}$. For $i > 0$, suppose that $\phi(\mathfrak{g}^{(i)}) = \mathfrak{h}^{(i)}$; we can then find that $\phi(\mathfrak{g}^{(i+1)}) = \phi([\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]) = [\phi(\mathfrak{g}^{(i)}), \phi(\mathfrak{g}^{(i)})] = [\mathfrak{h}^{(i)}, \mathfrak{h}^{(i)}] = \mathfrak{h}^{(i+1)}$. Note that the definition of a homomorphism is used in the second step. ☐

Proposition. *Suppose* $\mathfrak{q}$ *is a solvable ideal of a Lie algebra* $\mathfrak{g}$ *such that* $\mathfrak{g}/\mathfrak{q}$ *is solvable. Then* $\mathfrak{g}$ *is solvable as well.*

Proof. Let $(\mathfrak{g}/\mathfrak{q})^{(n)} = \mathbf{0}$ and consider the homomorphism $\pi : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{q}$. The existence of π is the canonical homomorphism assured from the standard homomorphism theorems. From the above proposition, we know that $\pi(\mathfrak{g}^{(n)}) = \mathbf{0}$ and so we can write $\mathfrak{g}^n \subset \mathfrak{q} = \ker \pi$. Now suppose $\mathfrak{q}^{(m)} = \mathbf{0}$ since $\mathfrak{q}$ is solvable. Then $\mathfrak{g}^{(n)}$ must be solvable (by the above proposition again) and, using the fact that $(\mathfrak{g}^{(i)})^{(j)} = \mathfrak{g}^{(i+j)}$, we can see $\mathfrak{g}^{(m+n)} = \mathbf{0}$. Thus, $\mathfrak{g}$ is solvable. ☐

Proposition. *Suppose* $\mathfrak{q}$ *and* $\mathfrak{r}$ *are solvable ideals of a Lie algebra* $\mathfrak{g}$*. Then* $\mathfrak{q}+\mathfrak{r}$ *is likewise solvable.*

Proof. The standard homomorphism theorems tell us there is a homomorphism between $(\mathfrak{q}+\mathfrak{r})/\mathfrak{r}$ and $\mathfrak{q}/(\mathfrak{q} \cup \mathfrak{r})$; let us call it ψ. Note that $\mathfrak{q}/(\mathfrak{q} \cup \mathfrak{r})$ is a homomorphic image of $\mathfrak{q}$, so it is solvable (by the first solvability prop.). Applying the first solvability prop. to ψ tells us that $(\mathfrak{q}+\mathfrak{r})/\mathfrak{r}$ is also solvable. Then the use of the second solvability prop. with $(\mathfrak{q}+\mathfrak{r})/\mathfrak{r}$ and $\mathfrak{r}$ shows that $\mathfrak{q}+\mathfrak{r}$ is solvable. ☐

Nilpotence propositions and proofs: (From Sec. 3.3.2 on page 8)

Proposition. *Suppose* $\mathfrak{g}$ *is a nilpotent Lie Algebra. Then all subalgebras and homomorphic images of* $\mathfrak{g}$ *are also nilpotent.*

Proof. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{h}^i \subset \mathfrak{g}^i$. Thus, $\mathfrak{h}$ is nilpotent. In addition, for an epimorphism $\phi : \mathfrak{g} \to \mathfrak{h}$, $\phi(\mathfrak{g}^{(i)}) = \mathfrak{h}^{(i)}$. We know that $\phi(\mathfrak{g}) = \mathfrak{h}$ by the definition of an epimorphism, and we can show the rest by induction: for $i > 0$, suppose that $\phi(\mathfrak{g}^i) = \mathfrak{h}^i$ and consider: $\phi(\mathfrak{g}^{i+1}) = \phi([\mathfrak{g}, \mathfrak{g}^i]) = [\phi(\mathfrak{g}), \phi(\mathfrak{g}^i)] = [\mathfrak{h}, \mathfrak{h}^i] = \mathfrak{h}^{i+1}$. ☐

Proposition. *If* $\mathfrak{g}/Z(\mathfrak{g})$ *is nilpotent, then* $\mathfrak{g}$ *is nilpotent as well.*

Proof. Suppose $\mathfrak{g}^n \subset Z(\mathfrak{g})$. Then $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subset [\mathfrak{g}, Z(\mathfrak{g})] = \mathbf{0}$. ☐

Proposition. *If* $\mathfrak{g}$ *is nilpotent and nonzero,* $Z(\mathfrak{g}) \neq \mathbf{0}$.

Proof. : Suppose $\mathfrak{g}^{n+1} = \mathbf{0}$. Thus, $[\mathfrak{g}, \mathfrak{g}^n] = \mathbf{0}$, which implies that $\mathfrak{g}^n = \{x \in \mathfrak{g} | [y, x] = 0 \, \forall y \in \mathfrak{g}\}$. This is the definition of the center. Note that $Z(\mathfrak{g})$ is the last nonzero term of the descending central series. ☐

APPENDIX B

We will show that Lie's Theorem (4.2) fails when $\mathbb{F}$ has prime characteristic. Suppose $\operatorname{char} \mathbb{F} = p$, where p is prime. Consider the $p \times p$ matrices:

$$x = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & \dots & p-2 & 0 \\ 0 & 0 & \dots & 0 & p-1 \end{pmatrix}$$

It is relatively simple to calculate

$$\begin{aligned} [x, y] &= xy - yx &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & p-1 \\ 0 = p & 0 & 0 & \dots & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & (p-2) \\ (p-1) & 0 & 0 & \dots & 0 \end{pmatrix} \\ & &= \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} = x. \end{aligned}$$

Note that $0 = p$ in the $1, p$ entry of xy because $\operatorname{char} \mathbb{F} = p$. Since $[x, y] = x$, we can conclude that x and y span a 2-dimensional solvable subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(p, \mathbb{F})$. By Lie's Theorem, x and y should have at least one common eigenvector. However, consider y. As it is a diagonal matrix, the eigenvector e_i corresponds to the eigenvalue $\lambda_i = i - 1$, $0 < i \leq p$. Thus, $ye_i = \lambda_i e_i$. If x shares a common eigenvector, then $xe_i = \rho e_i$ for some i, where ρ is an eigenvalue of x. Compute

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \lambda_i \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \lambda_{i-1} \\ \vdots \\ 0 \end{pmatrix}$$

so $xe_i = \lambda_{i-1}e_{i-1}$; x does not have a common eigenvector and Lie's Theorem fails. Thus, $\text{char}\,\mathbb{F}$ must be 0 whenever Lie's Theorem is applied.

Acknowledgement. This paper closely follows the development of Lie algebras used in Humphreys [4].

REFERENCES

[1] O'Connor, J.J.; Robertson, E. F. "Maruis Sophus Lie." The MacTutor History of Mathematics Archive. Accessed November 2, 2007. <http://www-history.mcs.st-andrews.ac.uk/Biographies/Lie.html>
[2] Belinfante, J. G; Kolman, B; Smith, H. A. *An Introduction to Lie Groups and Lie Algebras With Applications*. SIAM Review, 8 (1966), pp. 11-46.
[3] Howe, Roger. *Very Basic Lie Theory.* The American Mathematical Monthly, 90 (1983), pp. 600-623.
[4] Humphreys, James E. Introduction to Lie Algebras and Representation Theory. Springer, 1994.
[5] Cornwell, J. F. Group Theory in Physics: An Introduction. San Diego: Academic Press, 1997.
[6] Knapp, Anthony W. Lie Groups Beyond an Introduction. Boston: Birkhäuser, 2002.
[7] Miller, Willard. Symmetry Groups and their Applications. New York: Academic Press Inc, 1972.
[8] Erdmann, Karin; Wildon, Mark J. Introduction to Lie Algebras. Springer, 2006.
[9] Samelson, Hans. Notes on Lie Algebras. Springer, 1990.
[10] Onishchik, A. L.; Vinberg, Ernest Borisovich. Lie Groups and Lie Algebras III. Springer, 1994.
[11] Belinfante, J. G; Kolman, B; Smith, H. A. *An Introduction to Lie Groups and Lie Algebras, With Applications. II. The Basic Methods and Results of Representation Theory.* SIAM Review, 10 (1968), pp. 160-195.