Senior Seminar Proposal: Matchwebs
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ABSTRACT

A matchweb is a connected, minimal, normalized matching bipartite graph W(X, Y), where |X| = n, |Y| = k, and $n \leq k$. These matchwebs can be classified according to their core graphs Λ(X, Y*). Within the area of matchwebs and their cores, we consider the following question: Given an arbitrary positive integer n, where $n \leq k$, and n does not divide k, does there exist an (n, k) matchweb?

1. BACKGROUND

We construct a bipartite graph with n black vertices in the lower set and k white vertices in the upper set, where $n \leq k$. To ensure a matching, one sufficient condition is that the bipartite graph be normalized matching – that is, for each subset *C* of the upper set, the neighbourhood ΓC of the lower set for that subset satisfies $\frac{|C|}{k} \leq \frac{|\Gamma C|}{n}$, as depicted in Figure 1.

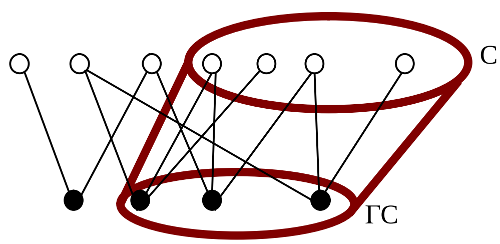


Figure 1: An example of normalized matching.

However, it isn't practical to check every subset of a bipartite graph for normalized matching, especially as that just ensures that *that* particular bipartite graph is normalized matching. Hence, to demonstrate a bipartite graph is normalized matching, we label each connection with a nonnegative integer such that for each white vertex, the sum of its connections is n, and for each black vertex, the sum for its connections is k, as seen in Figure 2.

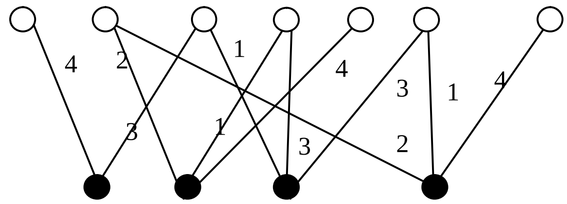


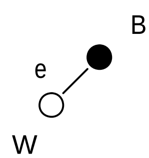
Figure 2: An illustration of integer labeling for normalized matching.

Since adding a connection to a normalized matching bipartite graph does not change the fact that it's normalized matching, we minimize the graph so that removing any connection causes the normalized matching condition to fail. Finally, in disjoint sections of a minimal, normalized matching bipartite graph, each section satisfies the minimal, normalized matching condition, so we consider the smallest instance – that is, a connected, minimal, normalized matching bipartite graph.

Hence, we define an (n, k) *matchweb* to be a connected minimal normalized matching bipartite graph. A matchweb is also a tree in which all leaves are in one set. We obtain a matchweb *core*, which may be interpreted as a simple hypertree, by removing all leaves from the top set. By a result of Hsu and Nolan, the labels of the edges of the core are determined mod n.

Hsu and Nolan went on to show a uniqueness theorem (that is, for each core, there is only one unique matchweb of form (n, k) – *if it exists*). Logan and I then used a labeling lemma (shown below) to prove the existence theorem – i.e. for every core $\Lambda(X, Y^*)$, there exist infinitely many associated matchwebs satisfying $k \equiv 1 \bmod n$.

Labeling lemma: Assume core Λ. What is the label on e?



Label on e is [kx], where x is the number of black vertices in component of $\Lambda - e$ containing B, and $[kx] \equiv kx \bmod n$, $[kx] \in \{1, \ldots, n\}$.

2. CURRENT WORK

The open problem I am considering is this:

Given an arbitrary positive integer n, where n < k, and n does not divide k, does there exist an (n, k) matchweb?

The specific cases I am working on are (n, k) matchwebs such that

(n, k) = (2d, 3d), where d = gcd(n, k),
= (n, 2n + 1), and
= (n, k ≡ (n – 1) mod n).

Thus far, I have found the restrictions for the existence of (n, 2n + 1) and (n, k ≡ (n – 1) mod n) matchwebs. First, for the (n, 2n + 1) case, as long as the maximum degree of a black vertex in the core is ≤ 3, we can construct an (n, 2n + 1) matchweb. Then, for the (n, k ≡ (n – 1) mod n) case, we have that the core graph produces a matchweb with k ≡ (n – 1) iff all white vertices in the core have degree 2. In fact, we have a 1-1 correspondence between matchwebs with k ≡ (n – 1) and a tree in which all leaves are white, all other white vertices have degree 2, and all black vertices have the same number of leaves.

3. FUTURE INTENTIONS

Besides producing a more complete and polished written account and explanation of the background material and my results, I plan to proceed with the case of (2d, 3d) matchwebs.

REFERENCES

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