Philosophy 2101 - Introduction to Symbolic Logic Lory Lemke

Set Theory and Russell's Paradox.

The basic expressions of Aristotelian logic are claims that relate two classes (or sets). It is quite a task to translate English sentences into the 4 categorical propositions of Aristotelian logic. This task assumes that every claim that can be true or false can be expressed as a relation between two sets and also that every subject and predicate of such a claim names a set. As you are now familiar with using Venn diagrams to illustrate sentences which relate two sets, you can appreciate the utility of using sets to reveal deductive relations. Afterall, sets can be naturally found, like the set of trees or they can be arbitrarily formed, like the following set: {my small nose, my red horseshoes, and your logic book}. However, it is difficult to describe exactly what a set is. We may also wonder whether any set exists apart from its members. For example, there can be empty sets, like the set of dollars in my wallet. Granted that I have no dollars in my wallet, so that this set is empty, does this set exist? If it doesn't then there are no empty sets, but if it does, then what is it since it simply illustrates the lack of anything in my wallet.

Worries about sets began when Bertrand Russell, in the first part of the 20^{th} century, discovered a paradox concerning sets. To appreciate this paradox, you will need to consider the fact that sets can be members of sets. For example, while a family is a set of individuals, we can construct a set F of families, where each member of the set F is also a set (of diverse individuals). We also can create an arbitrary set which has itself among its members, e.g. set L: {the Lemke family, snoopy the dog, set L}. The quality of "being a member of itself" is called reflexivity. Set L, as we have defined it, is reflexive, i.e., it contains itself. However, most sets are irreflexive, i.e., they are not members of themselves, e.g., the above defined set F and also the set D of all dogs: {Snoopy, etc.,}. Neither F nor D contains itself, or any other set for that manner.

This allows us to divide sets into two groups: those which are reflexive and those that are not. Consider the set I of all sets which are not members of themselves, i.e. irreflexive sets, e.g. set I: {set D, set F, etc,}, but not set L. Now, let's consider the question of whether I is a member of itself. If it is not, then by the definition of set I, it is in set I. But this is contradictory because we started with the hypothesis that I was not a member of itself. The only other alternative is for set I to be a member of itself, but that too is impossible because I is defined as the set which contains only those sets which are not members of themselves. As a result set I is not either reflexive or irreflexive!

The paradox can be summarized as follows: (1) it is true that every set is such that it is either a member of itself or it is not a member of itself; (2) set I, above, is neither. If (2) is true - and it certainly seems to be even if the reasoning for it is a bit complicated, then (1) is false. But (1) seems so simply true that it is hard to imagine how it could be false.