MATH 3401 (Ng/Spring 2009)
Handout 4
For class on **February 20-25, 2009.**

1. Defining the *Dual* problem from a given *Primal* problem.

Primal OR **Dual** (maximize)	**Dual** OR **Primal** (minimize)
constraints are:	variables are:
$\leq$	≥ 0
$\geq$	≤ 0
$=$	unrestricted
variables are:	constraints are:
≥ 0	$\geq$
≤ 0	$\leq$
unrestricted	$=$

2. Notes:

   a. The *dual* of the dual problem of (P) is equivalent to (P) itself.

   b. Recall that a linear programming problem (P) is *unbounded* if for every feasible solution there is another distinct feasible solution with a better objective value.

   c. Recall that a linear programming problem (P) is *optimal* if it is feasible and has a finite optimal solution value.

3. Relationships between *Primal* and *Dual* problems.

   a. (P) is optimal **if and only if** (D) is optimal. (Why?)

   b. **If** (P) is unbounded **then** (D) is infeasible. (Why?) (Converse is not true - why?).

   c. **If** (D) is unbounded **then** (P) is infeasible. (Why?) (Converse is not true - why?).

   d. **If** (P) is infeasible **then** (D) is either unbounded or infeasible. (Why?)

   e. **If** (D) is infeasible **then** (P) is either unbounded or infeasible. (Why?)

4. RESULT 1. **The Weak Duality Theorem**. For any primal feasible solution to (P) (max version), say, $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$, and for any dual feasible solution to (D), say, $(\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_m)$;

$$\sum_{j=1}^{n} c_j \bar{x}_j \leq \sum_{i=1}^{m} b_i \bar{y}_i$$

   In other words, the objective value of any primal feasible solution is at most that of any dual feasible solution. (Proof in class.)

5. RESULT 2. **The Strong Duality Theorem**. If the primal problem (P) has an optimal solution, say, $(x_1^*, x_2^*, \ldots, x_n^*)$, then the dual problem (D) has an optimal solution $(y_1^*, y_2^*, \ldots, y_m^*)$ such that

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

   In other words, if the primal problem has an optimal solution, then the dual problem must also have an optimal solution and these two solutions have the same objective function values.

6. RESULT 3. **Complementary Slackness Property**. (*Necessary and sufficient conditions for optimality*). Let $(x_1^*, x_2^*, \ldots, x_n^*)$ be a feasible solution to the primal problem (P) and let $(y_1^*, y_2^*, \ldots, y_m^*)$ be a feasible solution to the dual problem (D). Necessary and sufficient conditions for simultaneous optimality of $(x_1^*, x_2^*, \ldots, x_n^*)$ and $(y_1^*, y_2^*, \ldots, y_m^*)$ are

$$\sum_{i=1}^{m} a_{ij} y_i^* = c_j \text{ or } x_j^* = 0 \text{ (or both) for } j = 1, 2, \ldots, n$$

   and

$$\sum_{j=1}^{n} a_{ij} x_j^* = b_i \text{ or } y_i^* = 0 \text{ (or both) for } i = 1, 2, \ldots, m$$