Math 3401 (Ng/Spring 2009)
Handout 6 ©**Peh Ng**
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5 Some Fundamentals of Graph Theory

The purpose of this section is to establish some basic terminologies and concepts of *graph theory*, which are necessary for optimization problems on networks and graphs.

5.1 (Undirected) Graphs

A *graph*, denoted, $G = (V, E)$, is a pair (V, E) where V is a set of *vertices* and E is a set of two-element subsets of V called *edges*. ($E \subseteq \{(i, j) : i, j \in V\}$).

Definitions and Terminologies : Let $G = (V, E)$ be a given (undirected) graph.

- Let $u, v \in V$, where $u \neq v$, i.e. u, v are distinct vertices in V. Then u and v are said to be *adjacent* if the edge $(u, v) \in E$.
- An edge $(i, j) \in E$ is said to be *incident* to the vertices i and j.
- Let $v \in V$. An edge of the type (v, v) is called a *loop*.
- Let $u, v \in V$ where $u \neq v$. Then the edges (u, v) and (u, v) are called *parallel edges*.
- The *degree* of a vertex is the number of edges incident to the vertex, with a loop counting twice.
- A graph G is *simple* if it does NOT contain any loops nor parallel edges.

For examples of the aforementioned terms, refer to **Figure 1** and **Figure 2**.

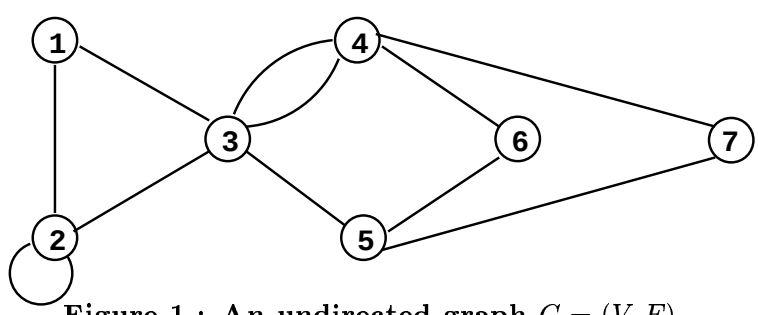

Figure 1 : An undirected graph $G = (V, E)$

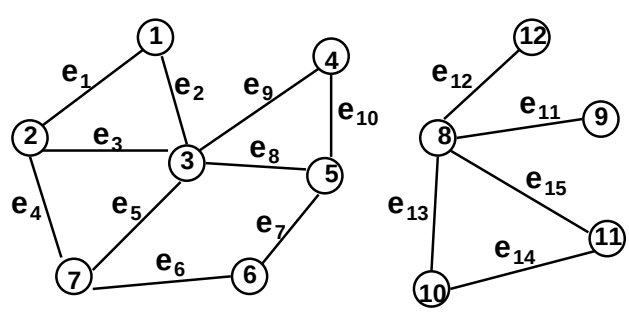

Figure 2 : A simple undirected graph $G = (V, E)$

Let $G = (V, E)$ be a graph.

- Let $\bar{V} \subseteq V$. Then a *subgraph of G induced by $\bar{V}$* is the graph whose vertex set is $\bar{V}$ and whose edge set is $\bar{E} \equiv \{(i,j) \in E : i, j \in \bar{V}\}$.
- Let $\tilde{E} \subseteq E$. Then a *subgraph of G induced by $\tilde{E}$* is the graph whose edge set is $\tilde{E}$ and whose vertex set is $\tilde{V} \equiv \{i, j \in V : (i,j) \in \tilde{E}\}$.
- A *spanning subgraph of G* is a subgraph of G (either induced by a vertex or edge set) that contains **all** vertices of G.

For example, in the graph of **Figure 2**, the subgraph induced by the vertex set $\bar{V} = \{1, 2, 3, 4, 6, 8, 10\}$ is given in **Figure 3**.

And the subgraph induced by the edge set $\tilde{E} = \{e_1, e_5, e_7, e_8, e_9, e_{11}\}$ is given in **Figure 4**.

Figure 3 : Subgraph, $\bar{G} = (\bar{V}, \bar{E})$

Figure 4 : Subgraph, $\tilde{G} = (\tilde{V}, \tilde{E})$

5.2 Some Structures of Graphs

Let $G = (V, E)$ be a graph.

Refer to **Figure 5** for an example.

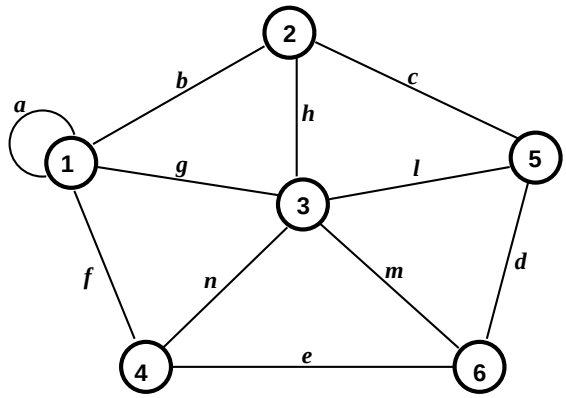

Figure 5 : $G = (V, E)$

- A <u>*walk from vertex v_0 to vertex v_k*</u> is a finite sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$, where each $v_i \in V$ for $i = 0, 1, 2, \ldots, k$ and each $e_i \equiv (v_{i-1}, v_i) \in E$ for $i = 1, 2, \ldots, k$.

  An example of a walk from vertex 4 to vertex 1 in **Figure 5** is $4, f, 1, a, 1, b, 2, h, 3, l, 5, c, 2, h, 3, g, 1$.

- A <u>*trail from vertex v_0 to vertex v_k*</u> is a walk from vertex v_0 to vertex v_k which contains NO repeated edges.

  An example of a trail from vertex 4 to vertex 1 in **Figure 5** is $4, f, 1, a, 1, b, 2, h, 3, l, 5, d, 6, m, 3, g, 1$.

- A <u>*path from vertex v_0 to vertex v_k*</u> is a trail from vertex v_0 to vertex v_k which contains NO repeated vertices.

  An example of a path from vertex 4 to vertex 1 in **Figure 5** is $4, e, 6, m, 3, g, 1$.

- A <u>*tour*</u> is a trail from vertex v_0 to vertex v_0.

- A <u>*cycle*</u> is a trail from vertex v_0 to vertex v_0 which contains NO other repeated vertices.

  An example of a cycle in **Figure 5** is $1, b, 2, h, 3, m, 6, e, 4, f, 1$.

A graph $G = (V, E)$, is said to be <u>connected</u> if there exists a path from vertex u to vertex v for all pairs of vertices $u, v \in V$.

Remark: Connectivity is an <u>equivalence relation</u> on the set of vertices V. WHY?
In other words, for $u, v, w \in V$, why are the following three conditions satisfied?
(i) u is connected to u,
(ii) if u is connected to v then v is connected to u,
(iii) if u is connected to v and if v is connected to w, then u is connected to w.

Corollary of Remark:

Given a graph $G = (V, E)$, there exists a partition of the vertex set V into non-empty subsets, say, $V_1, V_2, \ldots, V_\omega$ such that two vertices u and v are connected if and only if u and v belong to the same vertex set V_i.

The subgraphs induced by $V_1, V_2, \ldots, V_\omega$, i.e. $G[V_1]$, $G[V_2]$, ..., $G[V_\omega]$ are called the <u>*components*</u> of G. For examples of finding the components of a given graph $G = (V, E)$ see **Figure 6** and **Figure 7**.

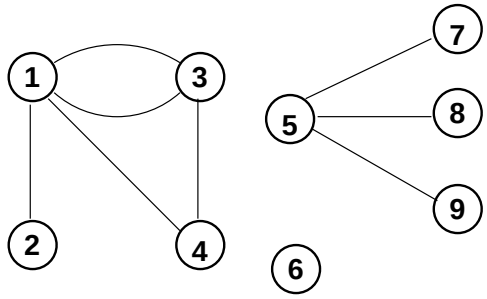

Figure 6 : $G = (V, E)$ **has more than one component**

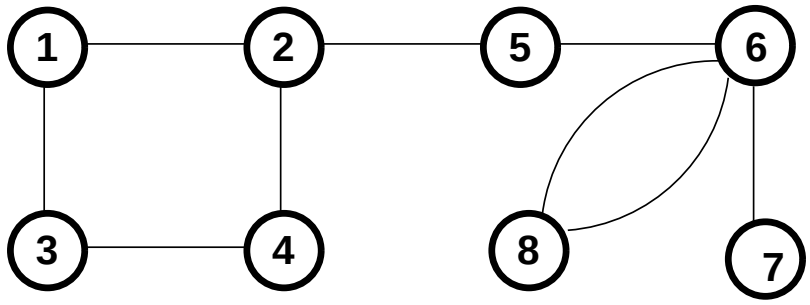

Figure 7 : $G = (V, E)$ **has exactly one component**

- A *forest* is a graph with no cycles.

- A *tree* is a forest that is connected.

- Let $G = (V, E)$ be a graph. A *spanning tree* of G is a spanning subgraph of G that is also a tree.

Theorem Let $G = (V, E)$ be a graph. Then G is connected if and only if G contains a spanning tree.

Theorem Let $T = (V, E)$ be a tree. Then T has exactly one more vertex than it has edges, i.e., $|V| = |E| + 1$.

Proof: Use mathematical induction on $|V|$.

5.3 Directed Graphs

A *directed graph*, denoted, $G = (V, A)$, is a pair (V, A) where V is a set of *vertices* and A is an ordered set of two-element subsets of V called *arcs*. ($A \subseteq \{(i, j) : i, j \in V\}$, and **note** that the arcs $(i, j) \neq (j, i)$).

For an example of a directed graph, refer to **Figure 8**.

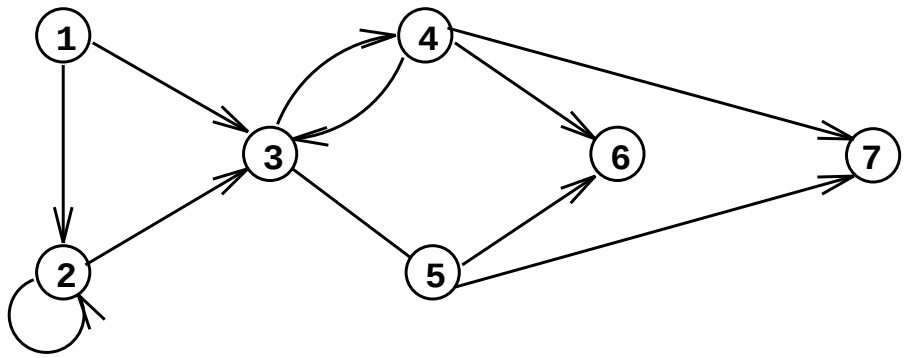

Figure 8 : A directed graph, $G = (V, A)$

Given a directed graph $G = (V, A)$, the *underlying graph associated with G*, is the undirected graph where the order on the arcs of A are ignored.

For instance, the undirected graph $G = (V, E)$ in **Figure 1** is the underlying graph of the directed graph $G = (V, A)$ in **Figure 8**.

5.4 Some structures of Directed Graphs

Let $G = (V, A)$ be a given directed graph.

- A *walk from vertex v_0 to vertex v_k* is a finite sequence $v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k$, where each $v_i \in V$ for $i = 0, 1, 2, \ldots, k$ and each $a_i \equiv (v_{i-1}, v_i) \in A$ or $a_i \equiv (v_i, v_{i-1}) \in A$ for $i = 1, 2, \ldots, k$.

- A *directed walk from vertex v_0 to vertex v_k* is a walk where each a_i is restricted to direction, i.e. $a_i \equiv (v_{i-1}, v_i) \in A$.

- A *trail from vertex v_0 to vertex v_k* is a walk from vertex v_0 to vertex v_k which contains NO repeated arcs

- A *directed trail from vertex v_0 to vertex v_k* is a directed walk from vertex v_0 to vertex v_k which contains NO repeated arcs

- A *path or a chain from vertex v_0 to vertex v_k* is a trail from vertex v_0 to vertex v_k which contains NO repeated vertices.

- A *directed path from vertex v_0 to vertex v_k* is a directed trail from vertex v_0 to vertex v_k which contains NO repeated vertices.

- A *cycle* is a trail from vertex v_0 to vertex v_0 which contains NO other repeated vertices.

- A *directed cycle or a circuit* is a directed trail from vertex v_0 to vertex v_0 which contains NO other repeated vertices.

Let $G = (V, A)$ be a directed graph.

- Let $u, v \in V$. Then vertex u is said to be *reachable* from vertex v if there exists a directed path from v to u in G.

- G is said to be *connected* if there exists a path from v to u **for every pair** of vertices $v, u \in V$.

- G is said to be *strongly connected* if u and v are reachable from each other, **for every pair** of vertices $v, u \in V$.