MATH 3411 (Ng/Fall 2009)
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6 Some Fundamentals of Graph Theory

The purpose of this section is to establish some basic terminologies and concepts of *graph theory*, which are necessary for applications of discrete mathematics on networks and graphs. For more details, see Ahuja et.al. [1], Bondy and Murty [2], and West [3].

6.1 (Undirected) Graphs

A *graph*, denoted, $G = (V, E)$, is a pair (V, E) where V is a set of *vertices* and E is a set of two-element subsets of V called *edges*. $(E \subseteq \{(i,j) : i, j \in V\})$.

Definitions and Terminologies : Let $G = (V, E)$ be a given (undirected) graph.

- Let $u, v \in V$, where $u \neq v$, i.e. u, v are distinct vertices in V. Then u and v are said to be *adjacent* if the edge $(u, v) \in E$.
- An edge $(i, j) \in E$ is said to be *incident* to the vertices i and j.
- Let $v \in V$. An edge of the type (v, v) is called a *loop*.
- Let $u, v \in V$ where $u \neq v$. Then the edges (u, v) and (u, v) are called *parallel edges*.
- The *degree* of a vertex is the number of edges incident to the vertex, with a loop counting twice.
- A graph G is *simple* if it does NOT contain any loops nor parallel edges.
- A simple graph $G = (V, E)$ with $|V| = n$ is called a *complete graph*, denoted K_n, if $E = \{(i,j) : i, j \in V\}$ i.e. there exists an edge incident to every pair of vertices.

For examples of the aforementioned terms, refer to **Figure 1** and **Figure 2**.

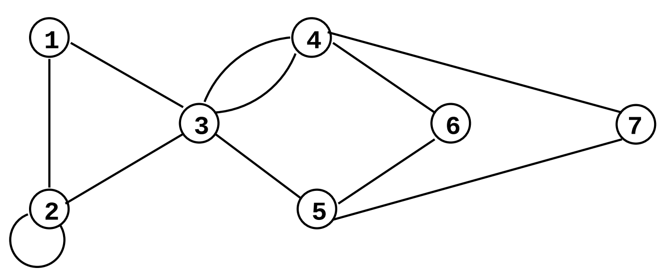

Figure 1 : An undirected graph $G = (V, E)$

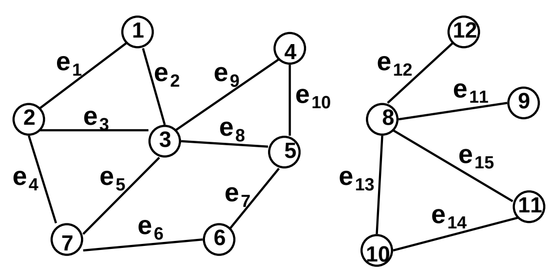

Figure 2 : A simple undirected graph $G = (V, E)$

Let $G = (V, E)$ be a graph.

- Let $\bar{V} \subseteq V$. Then a *subgraph of G induced by $\bar{V}$* is the graph whose vertex set is $\bar{V}$ and whose edge set is $\bar{E} \equiv \{(i,j) \in E : i, j \in \bar{V}\}$.
- Let $\tilde{E} \subseteq E$. Then a *subgraph of G induced by $\tilde{E}$* is the graph whose edge set is $\tilde{E}$ and whose vertex set is $\tilde{V} \equiv \{i, j \in V : (i,j) \in \tilde{E}\}$.
- A *spanning subgraph of G* is a subgraph of G (either induced by a vertex or edge set) that contains **all** vertices of G.

For example, in the graph of **Figure 2**, the subgraph induced by the vertex set $\bar{V} = \{1, 2, 3, 4, 6, 8, 10\}$ is given in **Figure 3**.
And the subgraph induced by the edge set $\tilde{E} = \{e_1, e_5, e_7, e_8, e_9, e_{11}\}$ is given in **Figure 4**.

Figure 3 : Subgraph, $\bar{G} = (\bar{V}, \bar{E})$

Figure 4 : Subgraph, $\tilde{G} = (\tilde{V}, \tilde{E})$

6.2 Some Structures of Graphs

Let $G = (V, E)$ be a graph.

Refer to **Figure 5** for an example.

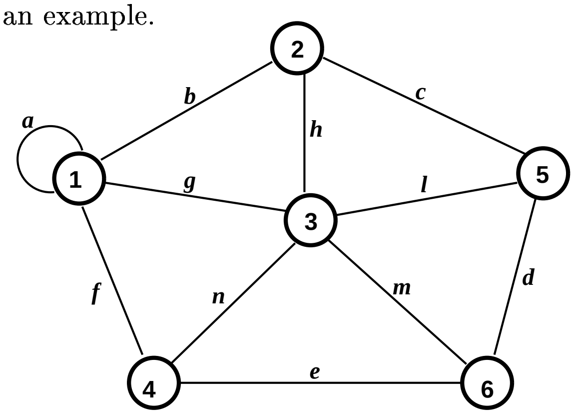

Figure 5 : $G = (V, E)$

- A <u>*walk from vertex v_0 to vertex v_k*</u> is a finite sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$, where each $v_i \in V$ for $i = 0, 1, 2, \ldots, k$ and each $e_i \equiv (v_{i-1}, v_i) \in E$ for $i = 1, 2, \ldots, k$.

  An example of a walk from vertex 4 to vertex 1 in **Figure 5** is $4, f, 1, a, 1, b, 2, h, 3, l, 5, c, 2, h, 3, g, 1$.

- A <u>*trail from vertex v_0 to vertex v_k*</u> is a walk from vertex v_0 to vertex v_k which contains NO repeated edges.

  An example of a trail from vertex 4 to vertex 1 in **Figure 5** is $4, f, 1, a, 1, b, 2, h, 3, l, 5, d, 6, m, 3, g, 1$.

- A <u>*path from vertex v_0 to vertex v_k*</u> is a trail from vertex v_0 to vertex v_k which contains NO repeated vertices.

  An example of a path from vertex 4 to vertex 1 in **Figure 5** is $4, e, 6, m, 3, g, 1$.

- A <u>*tour*</u> is a trail from vertex v_0 to vertex v_0.

- A <u>*cycle*</u> is a trail from vertex v_0 to vertex v_0 which contains NO other repeated vertices.

  An example of a cycle in **Figure 5** is $1, b, 2, h, 3, m, 6, e, 4, f, 1$.

A graph $G = (V, E)$, is said to be <u>connected</u> if there exists a path from vertex u to vertex v for all pairs of vertices $u, v \in V$.

Remark: Connectivity is an <u>equivalence relation</u> on the set of vertices V. WHY?
In other words, for $u, v, w \in V$, why are the following three conditions satisfied?
(i) u is connected to u,
(ii) if u is connected to v then v is connected to u,
(iii) if u is connected to v and if v is connected to w, then u is connected to w.

Corollary of Remark:

Given a graph $G = (V, E)$, there exists a partition of the vertex set V into non-empty subsets, say, $V_1, V_2, \ldots, V_\omega$ such that two vertices u and v are connected if and only if u and v belong to the same vertex set V_i.

The subgraphs induced by $V_1, V_2, \ldots, V_\omega$, i.e. $G[V_1]$, $G[V_2]$, ..., $G[V_\omega]$ are called the <u>*components*</u> of G. For examples of finding the components of a given graph $G = (V, E)$ see **Figure 6** and **Figure 7**.

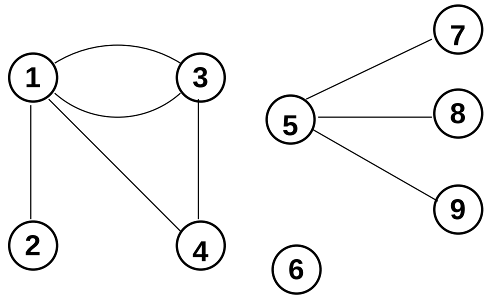

Figure 6 : $G = (V, E)$ **has more than one component**

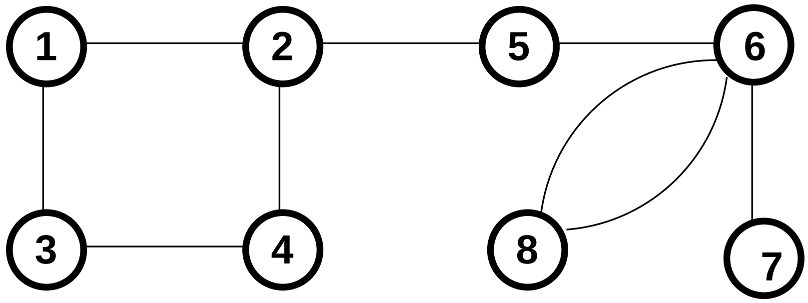

Figure 7 : $G = (V, E)$ **has exactly one component**

- A *forest* is a graph with no cycles.

- A *tree* is a forest that is connected.

- Let $G = (V, E)$ be a graph. A *spanning tree* of G is a spanning subgraph of G that is also a tree.

Theorem Let $G = (V, E)$ be a graph. Then G is connected if and only if G contains a spanning tree.

Theorem Let $T = (V, E)$ be a tree. Then T has exactly one more vertex than it has edges, i.e., $|V| = |E| + 1$.

HW: Prove this theorem by using mathematical induction on $|V|$.

6.3 Directed Graphs

A *directed graph*, denoted, $G = (V, A)$, is a pair (V, A) where V is a set of *vertices* and A is an ordered set of two-element subsets of V called *arcs*. ($A \subseteq \{(i, j) : i, j \in V\}$, and **note** that the arcs $(i, j) \neq (j, i)$).

For an example of a directed graph, refer to **Figure 8**.

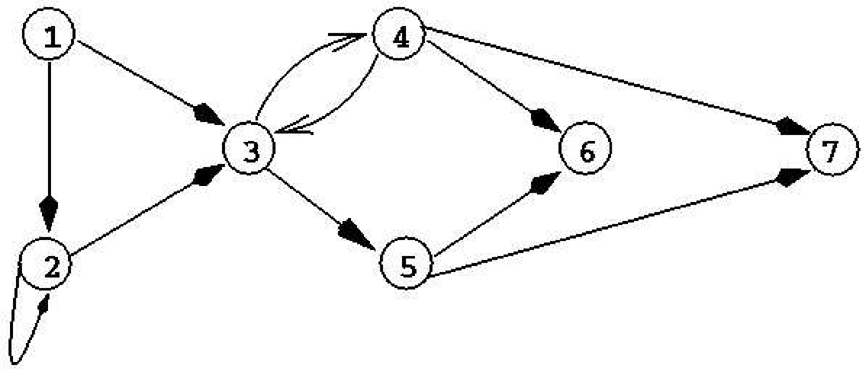

Figure 8 : A directed graph, $G = (V, A)$

Given a directed graph $G = (V, A)$, the *underlying graph associated with G*, is the undirected graph where the order on the arcs of A are ignored.

For instance, the undirected graph $G = (V, E)$ in **Figure 1** is the underlying graph of the directed graph $G = (V, A)$ in **Figure 8**.

6.4 Some structures of Directed Graphs

Let $G = (V, A)$ be a given directed graph.

- A *walk from vertex v_0 to vertex v_k* is a finite sequence $v_0, a_1, v_1, a_2, v_2, \ldots, a_k, v_k$, where each $v_i \in V$ for $i = 0, 1, 2, \ldots, k$ and each $a_i \equiv (v_{i-1}, v_i) \in A$ or $a_i \equiv (v_i, v_{i-1}) \in A$ for $i = 1, 2, \ldots, k$.

- A *directed walk from vertex v_0 to vertex v_k* is a walk where each a_i is restricted to direction, i.e. $a_i \equiv (v_{i-1}, v_i) \in A$.

- A *trail from vertex v_0 to vertex v_k* is a walk from vertex v_0 to vertex v_k which contains NO repeated arcs

- A *directed trail from vertex v_0 to vertex v_k* is a directed walk from vertex v_0 to vertex v_k which contains NO repeated arcs

- A *path or a chain from vertex v_0 to vertex v_k* is a trail from vertex v_0 to vertex v_k which contains NO repeated vertices.

- A *directed path from vertex v_0 to vertex v_k* is a directed trail from vertex v_0 to vertex v_k which contains NO repeated vertices.

- A *cycle* is a trail from vertex v_0 to vertex v_0 which contains NO other repeated vertices.

- A *directed cycle or a circuit* is a directed trail from vertex v_0 to vertex v_0 which contains NO other repeated vertices.

Let $G = (V, A)$ be a directed graph.

- Let $u, v \in V$. Then vertex u is said to be *reachable* from vertex v if there exists a directed path from v to u in G.
- G is said to be *connected* if there exists a path from v to u **for every pair** of vertices $v, u \in V$.
- G is said to be *strongly connected* if u and v are reachable from each other, **for every pair** of vertices $v, u \in V$.

Example of a connected but not strongly connected directed graph is the graph in **Figure 8**. Why is it not strongly connected?

Example of a strongly connected directed graph is given in **Figure 9**.

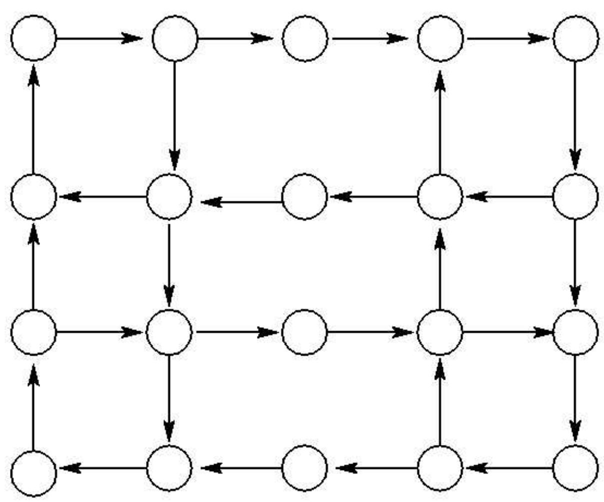

Figure 9 : A strongly connected directed graph, $G = (V, A)$

Why is it strongly connected?

6.5 Graph Isomorphisms

Let $G = (V, E)$ and $H = (\tilde{V}, \tilde{E})$ be simple graphs. Then, we say that G and H are **isomorphic**, denote $G \cong H$, if there exists two bijections $\theta : V \longrightarrow \tilde{V}$ and $\phi : E \longrightarrow \tilde{E}$ such that $e = (u, v) \in E$ if and only if $\phi(e) = (\theta(u), \theta(v))$.

The pair of bijections, (θ, ϕ), is called an **isomorphism** between G and H.

Note: Let R be a relation on a set of simple graphs defined by $(G, H) \in R$ if $G \cong H$. Is R (thus, isomorphism) an equivalence relation on a set of simple graphs?

For instance, the graphs G and H given in **Figure 10** are isomorphic to one another. **Why?**

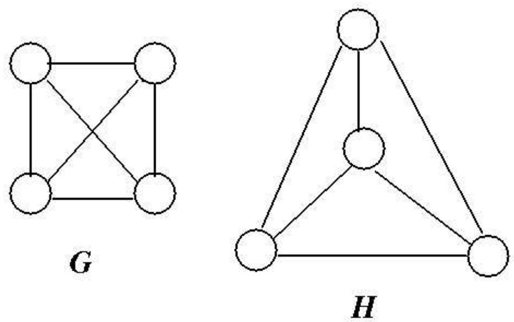

Figure 10 : Isomorphic G and H

Can you tell which graphs in **Figure 11** are isomorphic to one another?

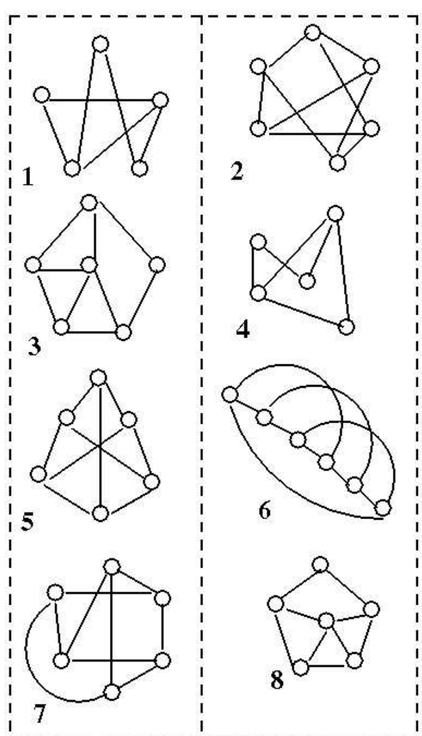

Figure 11 : Which of these graphs are isomorphic?

6.6 Matrix Representations of Graphs

6.6.1 Undirected Graphs

In addition to the representation of graphs by way of graphical circles (vertices) and curves (edges), there is an alternate but equivalent way to representing graphs, namely, *matrix representations.*

Although the graphical way of representing the structure of a graph is very useful in terms of visualization, this representation and its visualization become unmanageable when the size of the graph is large. By and large, *real-world problems* which are solved using *optimization models on graphs* tend to be of size $|V| \geq 500$. Thus, matrix representation of graphs becomes a more realistic tool to use than graphical representation. Indeed, this is the choice used by practitioners of applied discrete mathematics when solving problems computationally.

Let $G = (V, E)$ be any given graph where $|V| = m$ and $|E| = n$. Then:

1. the **vertex-edge incidence matrix of** G is the $m \times n$ matrix, denoted $\mathbf{M} = [m_{v,e}]$, whose rows are indexed by the set of vertices, whose columns are indexed by the set of edges and whose entries are members of $\{0, 1, 2\}$, such that $m_{v,e}$ is the number of times that edge $e \in E$ is incident to vertex $v \in V$. **Note that:**

   (a) for each vertex $v \in V$, the sum of the entries in every column of $\mathbf{M}$ corresponding to row v give $d_G(v)$.

   (b) if G is restricted to graphs without loops then the entries of its vertex-edge incidence matrix are 0 or 1.

2. the **adjacency incidence matrix of** G is the $m \times m$ matrix, denoted $\mathbf{A} = [a_{u,v}]$, whose rows and columns are indexed by the set of vertices and whose entries are non-negative integers, such that $a_{u,v}$ is the number of edges that are incident to u and v, for $u, v \in V$. **Note that:**

   (a) the matrix $\mathbf{A}$ is a square matrix that is also symmetric, namely, $A^T = A$.

   (b) if G is restricted to simple graphs (no loops and no parallel edges) then the entries of its adjacency matrix are 0 or 1.

Figure 12 shows two graphs, G and H. What are their vertex-edge incidence matrices and their adjacency matrices?

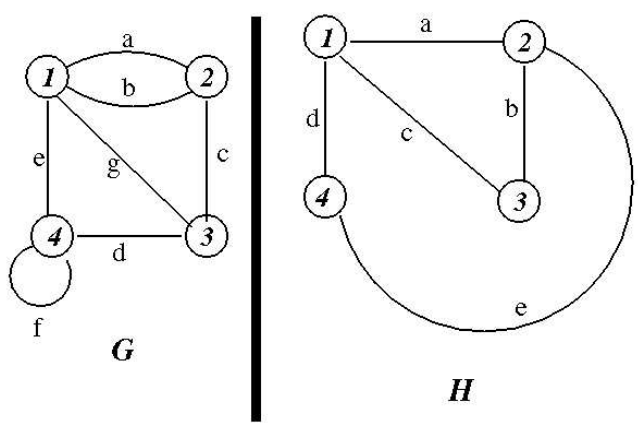

Figure 12 : Find the M and A of G **and of** H

6.6.2 Directed Graphs

When one is dealing with directed graphs, one can ask about matrix representations of their underlying undirected graphs, just as we did when we discussed graph structures in previous sections. In this case, the matrix representations of the underlying undirected graphs are exactly those discussed in the previous subsection.

In the world of (strictly) simple directed graphs, the conventional matrix representation is called **vertex-arc incidence matrix**.

Let $G = (V, A)$ be any given directed graph where $|V| = m$ and $|A| = n$. Then the **vertex-arc incidence matrix of** G is the $m \times n$ matrix, denoted $M = [m_{v,e}]$, whose rows are indexed by the set of vertices, whose columns are indexed by the set of arcs and whose entries are members of $\{0, \pm 1\}$, such that

$$m_{v,e} = \begin{cases} 1 & \text{if } e = (i, v) \text{ for some } i \in V, \text{ i.e. if } v \text{ is the head of } e \\ -1 & \text{if } e = (v, j) \text{ for some } j \in V, \text{ i.e. if } v \text{ is the tail of } e \\ 0 & \text{otherwise.} \end{cases}$$

The usage of $+1$ and -1 is either as specified above or interchanged meaning $+1$ and -1 is used if v is the tail and head of arc e, respectively. There is a arguably good reason for such flexibility and that is it all depends on how the context of *network flows* is presented in the discussion. (We will cover network flows in Sections ≥ 8.)

Figure 13 shows a directed graph $G = (V, A)$. What is its vertex-arc incidence matrix?

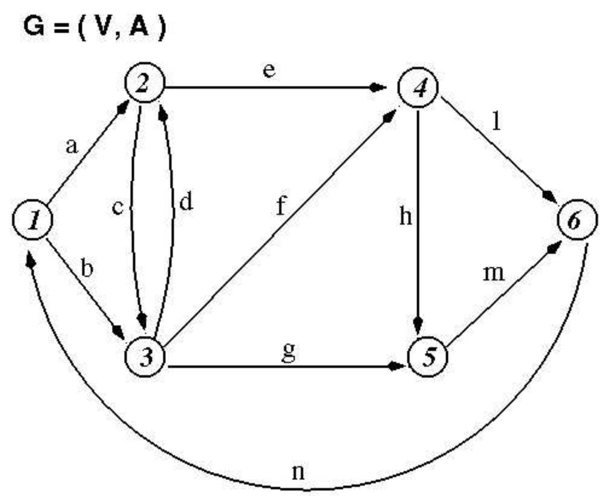

Figure 13 : Find the M of G

6.7 References

1. Ahuja, R.K., T.L. Magnanti and J.B. Orlin [1993], *Network Flows: Theory, Algorithms, and Applications*, Prentice Hall.

2. Bondy, J.A. and U.S.R. Murty [1976], *Graph Theory with Applications*, North-Holland.

3. West, D.B. [2001], *Introduction to Graph Theory* (second edition), Prentice Hall.